Chapter 5

Vorticity and Deformation

5.1 Derivation

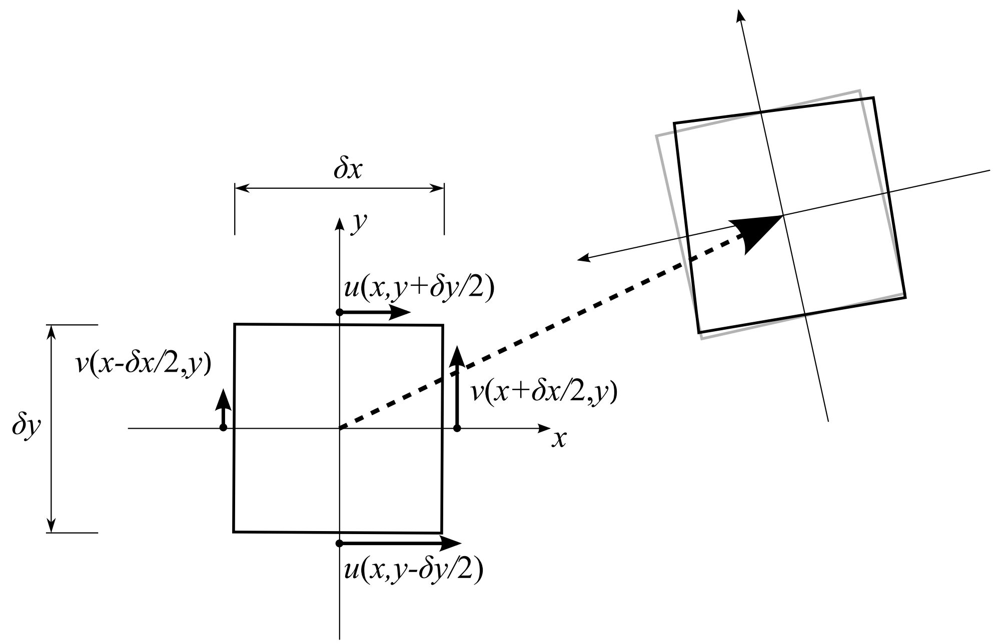

Figure 5.1: Diagram of the translation and rotation of a fluid element in the xy plane.

Consider Fig 5.1. This shows an element of fluid moving in two dimensions in a general fluid flow. The mean fluid velocity over the element determines its overall flow. *Differences* in velocity at different points on the element determine how its shape changes. If we define

$$\mathrm{d}x(x', y') = [u(x', y') - u(x, y)]\,\mathrm{d}t \tag{5.1}$$

$$\mathrm{d}y(x', y') = [v(x', y') - v(x, y)]\,\mathrm{d}t \tag{5.2}$$

so, for example, if the point x', y' is moving more slowly along the x axis than the refernce point, dx is negative.

We can say that the change in x and y relative to the centre, at the midpoint of each face is, for small dt

$$\mathrm{d}x(x + \delta x/2, y) = [u(x + \delta x/2, y) - u(x, y)]\,\mathrm{d}t \tag{5.3}$$
$$\mathrm{d}y(x + \delta x/2, y) = [v(x + \delta x/2, y) - v(x, y)]\,\mathrm{d}t \tag{5.4}$$
$$\mathrm{d}x(x - \delta x/2, y) = [u(x - \delta x/2, y) - u(x, y)]\,\mathrm{d}t \tag{5.5}$$
$$\mathrm{d}y(x - \delta x/2, y) = [v(x - \delta x/2, y) - v(x, y)]\,\mathrm{d}t \tag{5.6}$$
$$\mathrm{d}x(x, y + \delta y/2) = [u(x, y + \delta y/2) - u(x, y)]\,\mathrm{d}t \tag{5.7}$$
$$\mathrm{d}y(x, y + \delta y/2) = [v(x, y + \delta y/2) - v(x, y)]\,\mathrm{d}t \tag{5.8}$$
$$\mathrm{d}x(x, y - \delta y/2) = [u(x, y - \delta y/2) - u(x, y)]\,\mathrm{d}t \tag{5.9}$$
$$\mathrm{d}y(x, y - \delta y/2) = [v(x, y - \delta y/2) - v(x, y)]\,\mathrm{d}t \tag{5.10}$$

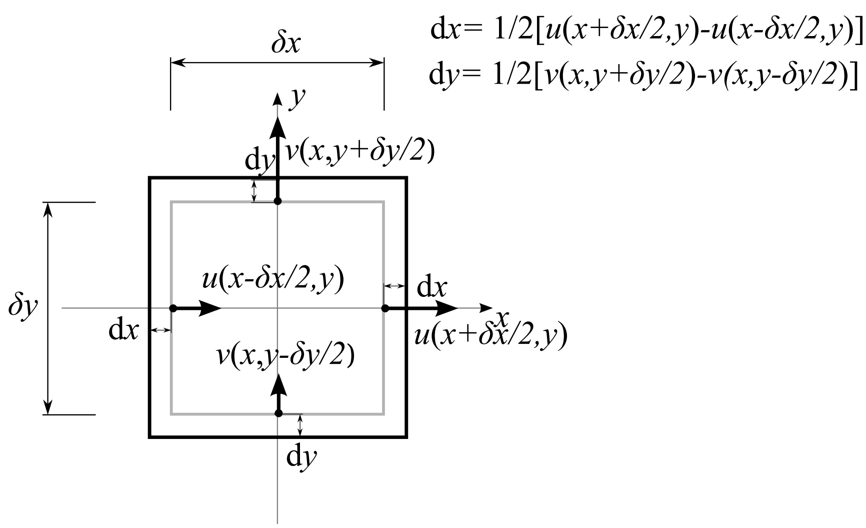

Figure 5.2: Schematic diagram of enlargement of the 'infinitesimal' volume δV located at $\mathbf{x}$ due to flow divergence.

Fig. 5.2 depicts pure divergence of the flow - stretching along the x axis and along the

y axis. The area changes from $\alpha = \delta x \delta y$ to

$$\begin{aligned}\alpha\,(t + \mathrm{d}t) = & (\delta x + \mathrm{d}x(x + \delta x/2, y) - \mathrm{d}x(x - \delta x/2, y)) \\ & \times (\delta y + \mathrm{d}y(x, y + \delta y/2) - \mathrm{d}y(x, y - \delta y/2)) \\ = & (\delta x + u(x + \delta x/2, y)\mathrm{d}t - u(x - \delta x/2, y)\mathrm{d}t) \\ & \times (\delta y + v(x, y + \delta y/2)\mathrm{d}t - v(x, y - \delta y/2)\mathrm{d}t) \\ = & \delta x \delta y + \delta y\,(u(x + \delta x/2, y) - u(x - \delta x/2, y))\,\mathrm{d}t \\ & + \delta x\,(v(x, y + \delta y/2) - v(x, y - \delta y/2))\,\mathrm{d}t \\ & + (u(x + \delta x/2, y) - u(x - \delta x/2, y))) \\ & \times (v(x, y + \delta y/2) - v(x, y - \delta y/2))\,\mathrm{d}t^2\end{aligned} \tag{5.11}$$

If we subtract $\alpha = \delta x \delta y$ from both sides and divide by $\delta x \delta y \mathrm{d}t$ we obtain

$$\begin{aligned}\frac{\alpha\,(t + \mathrm{d}t) - \alpha\,(t)}{\alpha \mathrm{d}t} = & \frac{(u(x + \delta x/2, y) - u(x - \delta x/2, y))}{\delta x} + \\ & \frac{(v(x, y + \delta y/2) - v(x, y - \delta y/2))}{\delta y} + \\ & \frac{(u(x + \delta x/2, y) - u(x - \delta x/2, y))\,(v(x, y + \delta y/2) - v(x, y - \delta y/2))}{\delta x \delta y}\mathrm{d}t\end{aligned} \tag{5.12}$$

Taking the limit $\delta x \to 0$, $\delta y \to 0$ and $\mathrm{d}t \to 0$, we obtain:

$$\frac{1}{\alpha}\frac{\mathrm{D}\alpha}{\mathrm{D}t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \mathbf{u} \tag{5.13}$$

Fig. 5.3 depicts pure rotation of our volume element about the z axis. Recall that the length of a arc of a circle radius r corresponding to angle θ in radians is $r\theta$. For small distances such as $\mathrm{d}x$ we can show that the angle subtended at x in Fig. 5.3 is such that $\frac{\delta x}{2}\delta\theta = \mathrm{d}x$.

Thus for a small anti-clockwise rotation $\delta\theta$ we can write:

$$\delta\theta(x + \delta x/2, y) = \mathrm{d}y(x + \delta x/2, y)/(\delta x/2) \tag{5.14}$$

$$\delta\theta(x - \delta x/2, y) = -\,\mathrm{d}y(x - \delta x/2, y)/(\delta x/2) \tag{5.15}$$

$$\delta\theta(x, y + \delta y/2) = -\,\mathrm{d}x(x, y + \delta y/2)/(\delta y/2) \tag{5.16}$$

$$\delta\theta(x, y - \delta y/2) = \mathrm{d}x(x, y - \delta y/2)/(\delta y/2) \tag{5.17}$$

Note the signs: at $x - \delta x/2, y$, y will be less that that at x, y, so $\mathrm{d}y < 0$ if $v(x - \delta x/2, y) < v(x, y)$, so we need the negative of $\mathrm{d}y$ for positive $\delta\theta$. Likewise at $x, y + \delta y/2$, $x$will be less that that at x, y, so $\mathrm{d}x < 0$ if $u(x, y + \delta y/2) < u(x, y)$, so we need the negative of $\mathrm{d}x$ for positive $\delta\theta$.

In pure rotation, these will be equal, but in general the fluid may be translating and deforming at the same time. We can define the rotation of the element by taking the

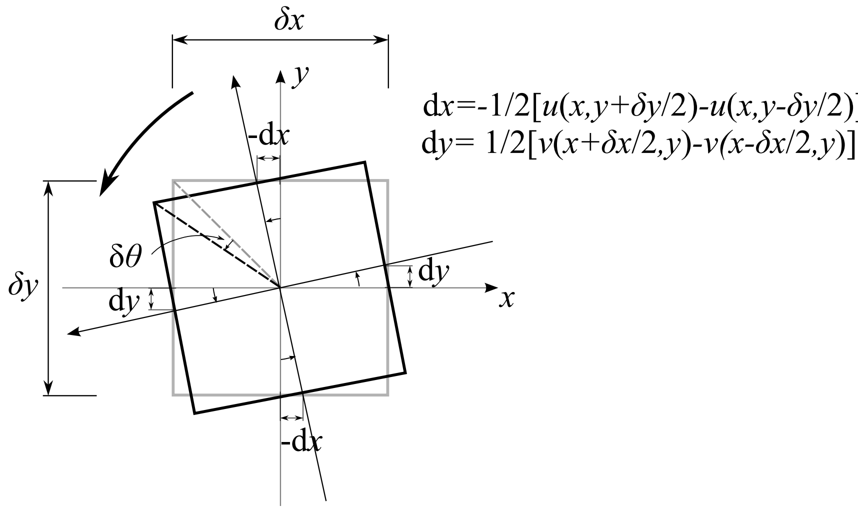

Figure 5.3: Schematic diagram of rotation about the z direction on the 'infinitesimal' volume δV located at $\mathbf{x}$.

average of these:

$$\begin{aligned}
\delta\theta(x,y) =& \frac{1}{4}\left[\mathrm{d}y(x+\delta x/2,y)/(\delta x/2) - \mathrm{d}y(x-\delta x/2,y)/(\delta x/2)\right.\\
&\left. -\mathrm{d}x(x,y+\delta y/2)/(\delta y/2) + \mathrm{d}x(x,y-\delta y/2)/(\delta y/2)\right]\\
=& \frac{1}{4}\left[\frac{v(x+\delta x/2,y)-v(x-\delta x/2,y)}{(\delta x/2)} - \frac{u(x,y+\delta y/2)-u(x,y-\delta y/2)}{(\delta y/2)}\right]\mathrm{d}t\\
=& \frac{1}{2}\left[\frac{v(x+\delta x/2,y)-v(x-\delta x/2,y)}{\delta x} - \frac{u(x,y+\delta y/2)-u(x,y-\delta y/2)}{\delta y}\right]\mathrm{d}t
\end{aligned}$$

Dividing through by $\mathrm{d}t$ and taking the limit $\delta x \to 0$, $\delta y \to 0$ and $\mathrm{d}t \to 0$, we obtain:

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{5.18}$$

This is the rate of rotation about the z axis - equivalent expressions for rotation about the x and y axes just by cyclicaly permuting the axes.

We define the x, y and z component of the *vorticity* as

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \tag{5.19}$$

$$\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \tag{5.20}$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{5.21}$$

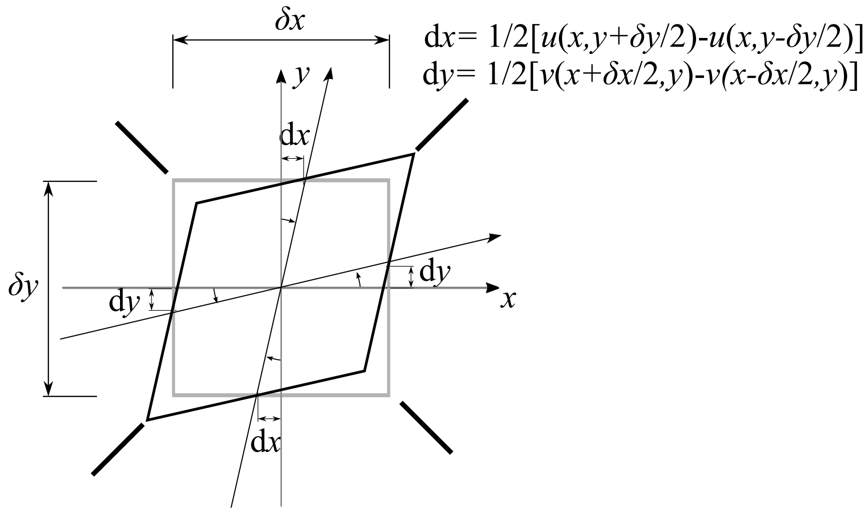

Figure 5.4: Schematic diagram of deformation about the z direction on the 'infinitesimal' volume δV located at $\mathbf{x}$.

We can also have deformation without a change of area (in 2D) or rotation. In a pure stretching deformation, without any rotation, the angle of rotation along one axis must be equal and opposite to that along the orthogonal axis so that the rotation diagnosed above is zero. Fig. 5.4 depicts pure deformation of our volume element with stretching along the $y = x$ line and compression along the $y = -x$. In a sense, the y axis is rotating in the opposite direction to the x axis. We can define an average deformation angle by :

$$\begin{aligned}\delta\theta(x,y) =& \frac{1}{4}\left[\mathrm{d}y(x+\delta x/2,y)/(\delta x/2) - \mathrm{d}y(x-\delta x/2,y)/(\delta x/2)\right.\\ &\left.+\mathrm{d}x(x,y+\delta y/2)/(\delta y/2) - \mathrm{d}x(x,y-\delta y/2)/(\delta y/2)\right]\\ =& \frac{1}{4}\left[\frac{v(x+\delta x/2,y)-v(x-\delta x/2,y)}{(\delta x/2)} + \frac{u(x,y+\delta y/2)-u(x,y-\delta y/2)}{(\delta y/2)}\right]\mathrm{d}t\\ =& \frac{1}{2}\left[\frac{v(x+\delta x/2,y)-v(x-\delta x/2,y)}{\delta x} + \frac{u(x,y+\delta y/2)-u(x,y-\delta y/2)}{\delta y}\right]\mathrm{d}t\end{aligned}$$

This is the rate deformation in the xy plane. We call this the *rate of strain tensor*, $e_{xy} = e_{yx}$. Dividing through by $\mathrm{d}t$ and taking the limit $\delta x \to 0$, $\delta y \to 0$ and $\mathrm{d}t \to 0$, we obtain:

$$e_{xy} = e_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{5.22}$$

We can define the rate of strain tensor, $e_{ij} = e_{ji}$, where $i, j \in \{1, 2, 3\}$ correspond to the directions x, y, z, by

$$e_{ij} = e_{ji} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{5.23}$$

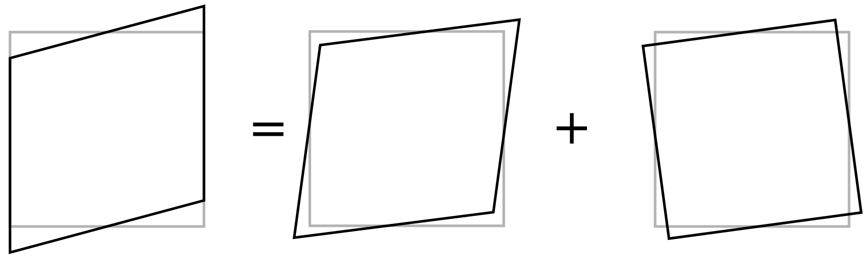

Figure 5.5: Schematic diagram of shearing along the y axis, showing how it comprises both deformation and rotation.

We often think os shear as just deforming the fluid. However, Fig 5.5 illustrates that a shearing motion in the y direction thus actually contains a component of rotation. This is effectively offset by shearing motion in the x direction along the $\delta x \delta y$ surfaces.

Mathematically, we can write:

$$\frac{\partial v}{\partial x} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = e_{xy} + \frac{1}{2}\omega_z \tag{5.24}$$

Consider the wind near the surface. A positive shear, $\frac{\partial u}{\partial z}$, is normal due to surface friction. This contains both strain of the fluid ($e_{xz} = e_{zx}$) and vorticity, ω_y.