Chapter 2

Taylor-Maclaurin Series and the exponential function

2.1 Maclaurin Series

Suppose we could express a function $y(x)$ as a (possibly infinite) polynomial of x:

$$y(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... \equiv \sum_0^\infty a_n x^n \tag{2.1}$$

Clearly, if we set $x = 0$, then

$$a_0 = y(0) \tag{2.2}$$

If we differentiate then we obtain another series

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a_1 + 2a_2 x + +3a_3 x^2 ... + n a_n x^{n-1} + ... \equiv \sum_1^\infty n a_n x^{n-1} \tag{2.3}$$

If we again set $x = 0$, then

$$a_1 = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} \tag{2.4}$$

Differentiating again leads to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2a_2 + 3 \times 2a_3 x ... + n \times (n-1) a_n x^{n-2} + ... \equiv \sum_1^\infty n(n-1) a_n x^{n-2} \tag{2.5}$$

so

$$a_2 = \frac{1}{2} \left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0} \tag{2.6}$$

If we carry on with this procedure, we can state a general rule (for $n > 0$):

$$a_n = \frac{1}{n!} \left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^n}\right|_{x=0} \tag{2.7}$$

so

$$y(x) = y(0) + \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} x + \frac{1}{2}\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^n}\right|_{x=0} x^2 + ... + \frac{1}{n!}\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^n}\right|_{x=0} x^n + ... \tag{2.8}$$

$$\equiv y(0) + \sum_1^\infty \frac{1}{n!}\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^n}\right|_{x=0} x^n \tag{2.9}$$

The Maclaurin series for a polynomial is, of course, the polynomial itself, and is of finite length. Functions which are not polynomials therefore have infinite series expansions. In any particular case, the series may not converge, or may only converge for some range of x.

Using the notation

$$y^{(n)}(a) \equiv \left.\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right|_{x=a} \tag{2.10}$$

with $y^{(0)} \equiv y$, the Maclaurin series can be compactly written

$$y(x) = \sum_0^\infty \frac{1}{n!} y^{(n)}(0)\, x^n \tag{2.11}$$

2.2 Taylor Series

We can use the Maclaurin series to expand about any point $x = a$, using the simple substitution $x' = x - a$ so $x' = 0$ when $x = a$. Bearing in mind the chain rule, so

$$\frac{\mathrm{d}y}{\mathrm{d}x'} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}x'} = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{2.12}$$

we can quickly arrive at

$$y(x) = \sum_0^\infty \frac{1}{n!} y^{(n)}(a)\,(x-a)^n \tag{2.13}$$

Since the Taylor series is more general, and the Maclaurin series is included (with $a = 0$) we often refer to the Taylor-Maclaurin series or just the Taylor series as a general series expansion.

2.2.1 Examples

$y = \sin x$;

$y^{(1)} = \cos x$, $y^{(2)} = -\sin x$, $y^{(3)} = -\cos x$, $y^{(4)} = \sin x$. Thus, the pattern repeats itself every 4 terms. At $x = 0$, $y^{(0)} = 0$, $y^{(1)} = 1$, $y^{(2)} = 0$, $y^{(3)} = -1$. Thus:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ... = \sum_{n=1}^\infty (-1)^{(n+1)} \frac{1}{(2n+1)!} x^{(2n+1)} \tag{2.14}$$

$y = \cos x$;

$y^{(1)} = -\sin x$, $y^{(2)} = -\cos x$, $y^{(3)} = \sin x$, $y^{(4)} = \cos x$. Thus, the pattern repeats itself every 4 terms. At $x = 0$, $y^{(0)} = 1$, $y^{(1)} = 0$, $y^{(2)} = -1$, $y^{(3)} = 0$. Thus:

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ... = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{(2n)} \tag{2.15}$$

$y = (1+x)^{-1}$;

$y^{(1)} = -(1+x)^{-2}$, $y^{(2)} = 2(1+x)^{-3}$, $y^{(3)} = -3 \times 2(1+x)^{-4}$. Thus, the general pattern is $y^{(n)} = (-1)^n n! (1+x)^{-(n+1)}$. At $x = 0$, $y^{(n)}(0) = (-1)^n n!$. Thus:

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - ... = \sum_{n=0}^{\infty} (-1)^n x^n \tag{2.16}$$

2.3 The exponentional function

Exponential functions arise when the rate of change of a variable is proportional to how big the variable is. A classic example is radioactive decay - the number of atoms which decay in unit time is proportional to how many undecayed atoms we have:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky \tag{2.17}$$

Let us define a function $\exp(x)$ by the property that its first derivative is equal to itself:

$$\frac{\mathrm{d}}{\mathrm{d}x} \exp(x) = \exp(x) \tag{2.18}$$

This property makes the function incredibly useful, especially for solving differential equations.

The *natural logarithm* is the inverse function of the exponential function. If $y = \exp x$ then $x = \ln y$.

Suppose $y = \ln x$, then $x = \exp y$. Since $\frac{\mathrm{d}x}{\mathrm{d}y} = x$ we can say

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}. \tag{2.19}$$

Suppose $y = \ln(f(x) g(x))$. From the chain and product rules

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{fg}\left(g\frac{\mathrm{d}f}{\mathrm{d}x} + f\frac{\mathrm{d}g}{\mathrm{d}x}\right) \\ &= \frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}x} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\ln f + \frac{\mathrm{d}}{\mathrm{d}x}\ln g \end{aligned} \tag{2.20}$$

Thus, within a constant,

$$\ln(fg) = \ln f + \ln g. \tag{2.21}$$

It is easy to show that the constant must be zero. (Set $f = 1$ and/or $g = 1$, we must conclude that $\ln 1 = 0$). Thus, $\ln x$ does, indeed, have the property of logarithms.

Furthermore, taking the exponential function of both sides:

$$fg = \exp(\ln f + \ln g) \tag{2.22}$$

It follows that if $f = \exp(ax)$ and $g = \exp(bx)$,

$$\exp(ax)\exp(bx) = \exp(ax + bx) = \exp((a+b)x) \tag{2.23}$$

This is the property of functions of the form $y = c^x$.

Thus, the exponential function must equal some number, which we call e, raised to the power x.

The Maclaurin series of $\exp x$ is very easy to derive since every derivative equals $\exp x$ and $\exp 0 = 1$. Thus $y^{(n)}(0) = 1$ and

$$\exp x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + ... = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \tag{2.24}$$

It is easy to verify that this series satisfies eq. (2.18).

Furthermore, since $e^1 = e$:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + ... = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828... \tag{2.25}$$

2.3.1 Alternative definition of the exponential function

Note that there is an alternative definition of $\exp x$. We start from the idea that we inflate a quantity in proportion to how much of the quantity we have. Consider gaining interest on some sum of money P, with an interest rate x, so after a given period, say a year, P becomes $P(1 + x)$, with x the interest rate per year. However, this is only true if we apply the interest once per year. Suppose we apply it twice yearly. The rate would then be $x/2$ but the we would apply the factor after half a year, so P becomes $P(1 + x/2)$, then after a further half year we apply the same factor to the total, so over the year P becomes $P(1 + x/2)^2$. In general, if we apply the interest n times per year, after a year P becomes $P(1 + x/n)^n$. We can continue this argument to apply the interest continuously. Then P becomes $P \exp x$ where we define $\exp x$ by:

$$\exp x = \lim_{n\to\infty} (1 + x/n)^n \tag{2.26}$$

From this we can prove all the above properties. For example, using the chain rule, with $f = g^n$ and $g = (1 + x/n)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \exp x &= \lim_{n\to\infty} n\left(1 + x/n\right)^{n-1} \times \frac{1}{n} \\
&= \lim_{n\to\infty} \left(1 + x/n\right)^{n-1} \\
&= \lim_{n\to\infty} \left(1 + \frac{x}{n-1}\frac{n-1}{n}\right)^{n-1} \\
&= \lim_{n\to\infty} \left(1 + \frac{x}{n-1}\left(1 + \frac{1}{n}\right)\right)^{n-1} \\
&= \exp x
\end{aligned} \tag{2.27}$$

where the last step makes us of the fact that the term $\left(1 + \frac{1}{n}\right)$ clearly tends to 1 as n tends to infinity, and we can then change $n - 1$ to, say, $m = n - 1$ and take the limit m tends to infinity.

2.3.2 Relationship with complex numbers and the Argand diagram

We shall cover complex numbers in the next chapter, but for the moment, note that, if $i = \sqrt{-1}$, we can write, from the above:

$$\exp(ix) = 1 + ix + \frac{1}{2!}i^2x^2 + \frac{1}{3!}i^3x^3 + \frac{1}{4!}i^4x^4 + \frac{1}{5!}i^5x^5 + ... = \sum_{n=0}^{\infty} \frac{1}{n!}i^nx^n \tag{2.28}$$

Since we can simplify terms like $i^2 = -1$ and $i^4 = 1$, this separates into real and imaginary terms (i.e. terms containing i). If we make this separation we obtain:

$$\begin{aligned}
\exp(ix) = &\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ...\right) \\
&+ i \times \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ...\right)
\end{aligned} \tag{2.29}$$

Comparison with the series expansions above leads to the identity:

$$\exp(ix) = \cos x + i \sin x \tag{2.30}$$

If we plot a point in a 2D Cartesian coordinate system at (x, y) then we note (Fig. 2.1) that the vector joining the origin to that point makes an angle θ with the x axis and has length r, with:

$$x = r\cos\theta \tag{2.31}$$

$$y = r\sin\theta \tag{2.32}$$

so

$$r = \sqrt{(x^2 + y^2)} \tag{2.33}$$

$$\theta = \arctan \frac{y}{x} \tag{2.34}$$

Suppose we identify the point (x, y) with the complex number $z = x + iy$, then from eq. (2.30), we can also write

$$z = r \exp{(i\theta)} \equiv re^{i\theta} \tag{2.35}$$

. This is an extremely useful identity, as the next chapter will show. Note that the whole *complex plane* is spanned by $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$ or $\theta \in [-\pi, \pi)$.

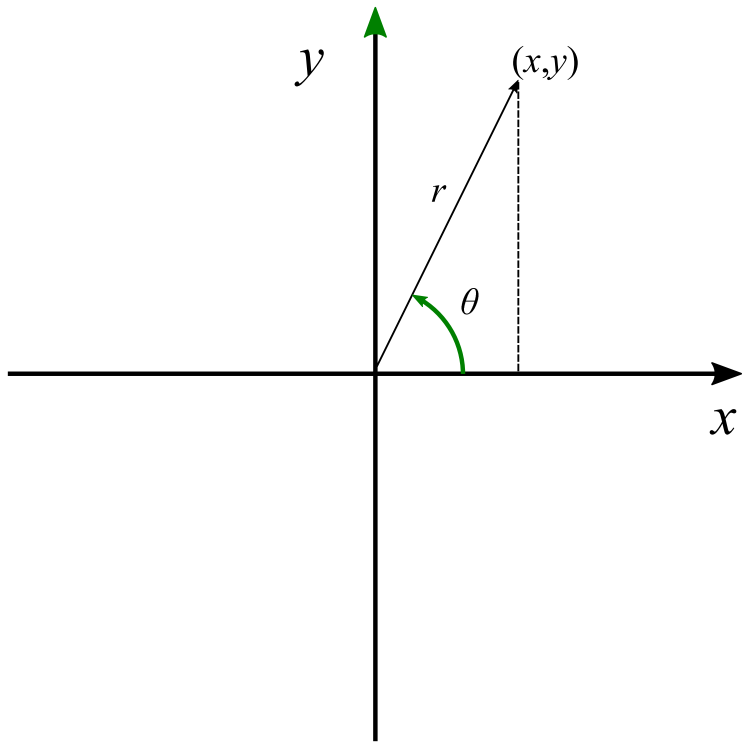

Figure 2.1: The Argand diagram.