Chapter 1

Differentiation

1.1 Fundamental definition of the derivative

Suppose you are recording the distance you have travelled on a journey; each distance s is recorded with the time t at which the distance was reached. So s is a function of time, $s(t)$. We estimate our speed, v, as follows:

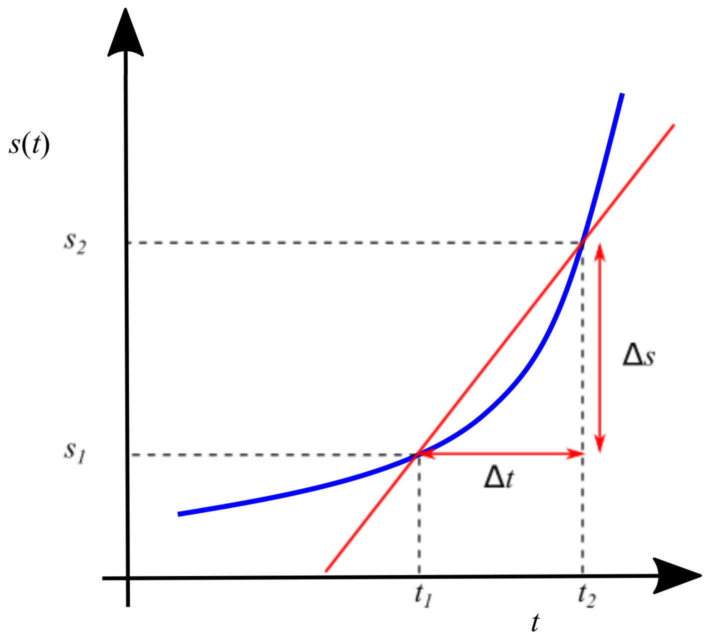

Figure 1.1: Calculating the rate of change.

$$v = \frac{s_2 - s_1}{t_2 - t_1} = \frac{\Delta s}{\Delta t} = \frac{s(t_1 + \Delta t) - s(t_1)}{\Delta t} \tag{1.1}$$

Note the three different, but equivalent, expressions. They represent an estimate of the mean speed over the time range t_1 to t_2.

We can make a better estimate of the speed at t_1 by moving t_2 closer to t_1 (Fig. 1.2).

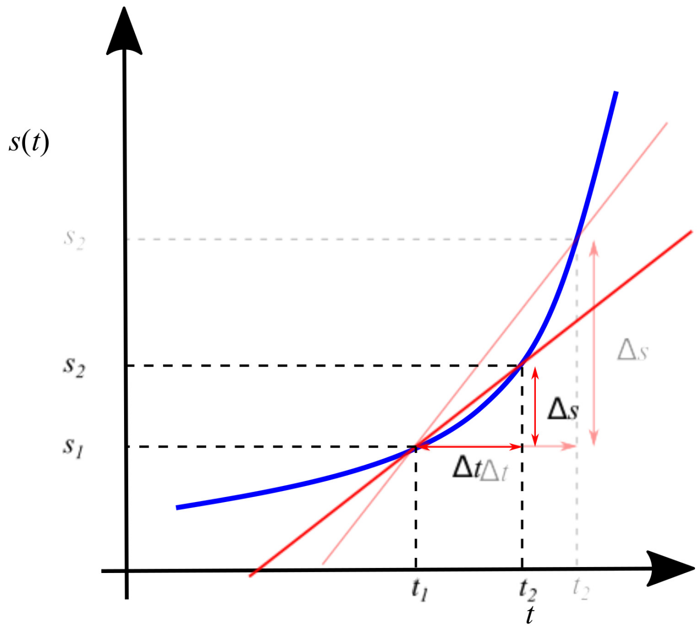

Figure 1.2: Calculating the rate of change more accurately.

As t_2 gets closer and closer to t_1, we can say that the estimated speed is a better and better estimate of the speed exactly at t_1. Eventually, the two points are so close together that the line through them does not cut the curve, but becomes tangential to it at t_1 (Fig. 1.3).

Thus, the speed at t_1 is the slope of the tangent at t_1, which we can calculate by taking the slope of two points separated by Δt in the limit that Δt tends to zero.

We define the gradient at t_1 by:

$$v\left(t_1\right) = \frac{\mathrm{d}s}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{s\left(t_1 + \delta t\right) - s\left(t_1\right)}{\delta t} \tag{1.2}$$

This is known as the 'first deriviative' or just the 'derivative' of s. The choice of letters in this definition is irrelevant, and you may be more familiar with:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{y\left(x + \delta x\right) - y\left(x\right)}{\delta x} \tag{1.3}$$

Note the slight change of notation: we have used a small increment of δt – conventionally, we use δ to signify our intention to take the limit to zero, while Δ signifies a finite increment. However, this is not a hard and fast rule, and, moreover, some texts use d rather than δ.

Not all functions are differentiable, and some are not differentiable everywhere. For example, the function

$$\begin{aligned} y = &-1\,; \qquad && x <= 0 \\ &+1\,; && x > 0 \end{aligned} \tag{1.4}$$

cannot be differentiated at $x = 0$ - the limit becomes $2/\delta x$. We often say that the derivative is infinite, which can be useful. Other functions are even less well-behaved - e.g. $y = x^{-1}$.

We often need to make it clear that a derivative is evaluated at a particular value of the independent variable, say $x = x_1$. In this case we write, for example

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_1} \tag{1.5}$$

or, more succinctly

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_1} \tag{1.6}$$

1.1.1 Approximation of functions

From the definition of the first derivative, a simple rearrangement gives us:

$$y\left(x + \delta x\right) = y\left(x\right) + \frac{\mathrm{d}y}{\mathrm{d}x}\delta x + O\left(\delta x^2\right) \tag{1.7}$$

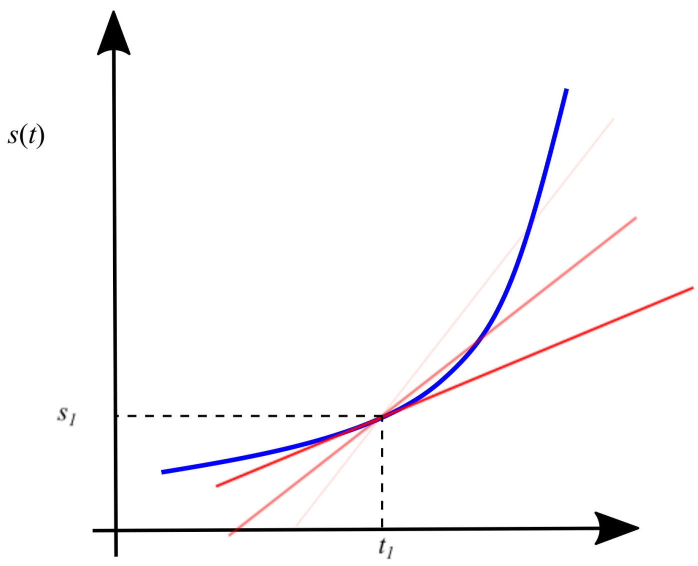

Figure 1.3: Calculating the rate of change at point 1.

1.2 Calculation of the derivative of analytic functions

Of course, if we just have a graph on paper, we can only estimate this approximately, but if we have a formula (or 'analytic function', we can often derive a formula for the first derivative. For example, suppose $s = kt^2$ where k is a constant. Then

$$\begin{aligned} s\left(t+\delta t\right) =& k\left(t+\delta t\right)^2 \\ =& k\left(t^2 + 2t\delta t + \delta t^2\right) \end{aligned} \tag{1.8}$$

so, in the above definition

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} =& \lim_{\delta t\to 0} \frac{s\left(t+\delta t\right) - s\left(t\right)}{\delta t} \\ =& \lim_{\delta t\to 0} \left[\frac{k\left(t^2 + 2t\delta t + \delta t^2\right) - kt^2}{\delta t}\right] \\ =& \lim_{\delta t\to 0} \left[\frac{2kt\delta t + k\delta t^2}{\delta t}\right] \\ =& \lim_{\delta t\to 0} \left[2kt + k\delta t\right] \\ =& 2kt \end{aligned} \tag{1.9}$$

We thus conclude (setting $k = 1$) that the first derivative of $s = t^2$ with respect to t is $\frac{\mathrm{d}s}{\mathrm{d}t} = 2t$.

Similarly, suppose $y = mx + c$ (a straight line), then

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} =& \lim_{\delta x\to 0} \frac{y\left(x+\delta x\right) - y\left(x\right)}{\delta x} \\ =& \lim_{\delta x\to 0} \left[\frac{m\left(x+\delta x\right) + c - \left(mx + c\right)}{\delta x}\right] \\ =& \lim_{\delta x\to 0} \left[\frac{m\delta x}{\delta x}\right] \\ =& \lim_{\delta x\to 0} m \\ =& m \end{aligned} \tag{1.10}$$

which is just the gradient of the straight line. Thus (setting $m = 1$) , the first derivative of $y = x$ with respect to x is just 1 and the first derivative of a constant is zero, as expected.

Finally, suppose $y = x^{-1}$, then

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{y\left(x+\delta x\right) - y\left(x\right)}{\delta x} \\
&= \lim_{\delta x \to 0} \left[\frac{\left(x+\delta x\right)^{-1} - x^{-1}}{\delta x}\right] \\
&= \lim_{\delta x \to 0} \left[\frac{\frac{x-(x+\delta x)}{(x+\delta x)x}}{\delta x}\right] \\
&= \lim_{\delta x \to 0} \left[\frac{\frac{-\delta x}{(x^2+x\delta x)}}{\delta x}\right] \\
&= \lim_{\delta x \to 0} \left[\frac{-1}{(x^2+x\delta x)}\right] \\
&= \frac{-1}{x^2} = -x^{-2} \qquad (1.11)
\end{aligned}$$

Note that, while the process of taking the limit $\delta x \to 0$ may seem difficult, in practice it is very straightforward.

1.2.1 Derivatives of trigonometric functions

To derive the derivatives of trigonometric functions we need to use some trigonometric identities:

$$\sin\left(A+B\right) = \sin A \cos B + \cos A \sin B \qquad (1.12)$$

$$\cos\left(A+B\right) = \cos A \cos B - \sin A \sin B \qquad (1.13)$$

plus the facts that $\lim_{x\to 0} \sin x = x$ and $\lim_{x \to 0} \cos x = 1$

Using these, if $y\left(x\right) = \sin x$, then

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{y\left(x+\delta x\right) - y\left(x\right)}{\delta x} \\
&= \lim_{\delta x \to 0} \left[\frac{\sin\left(x+\delta x\right) - \sin x}{\delta x}\right] \\
&= \lim_{\delta x \to 0} \left[\frac{\sin x \cos \delta x + \cos x \sin \delta x - \sin x}{\delta x}\right] \\
&= \lim_{\delta x \to 0} \left[\frac{\cos x \delta x}{\delta x}\right] \\
&= \cos x \qquad (1.14)
\end{aligned}$$

and if $y(x) = \cos x$, then

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{\delta x} \\
&= \lim_{\delta x \to 0} \left[\frac{\cos(x+\delta x) - \cos x}{\delta x}\right] \\
&= \lim_{\delta x \to 0} \left[\frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos x}{\delta x}\right] \\
&= \lim_{\delta x \to 0} \left[\frac{-\sin x \delta x}{\delta x}\right] \\
&= -\sin x
\end{aligned} \tag{1.15}$$

1.2.2 Notation

We have derived the derivative, e.g. $\frac{\mathrm{d}y}{\mathrm{d}x}$, given y as a function of x. However, we often cut corners by not identifying the function explicity, for example by writing:

$$\frac{\mathrm{d}x^2}{\mathrm{d}x} = 2x \tag{1.16}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 4x + 3\right) = 2x + 4 \tag{1.17}$$

1.3 Repeated differentiation

The first derivative is a function itself. So we can write the second derivative as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{1.18}$$

and so on. This notation is somewhat cumbersome, and a more compact notation is used:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{1.19}$$

Note the position of the superscripts - think of this as the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ applied twice. In general, we write $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$ as the nth derivative.

1.3.1 More notation

Sometimes even $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$ is long-winded, and the first derivative is just written y', the second y'' etc.. Alternatively $y^{(1)}$, $y^{(2)}$,....$y^{(n)}$. When using these notations, it should be made very clear what is intended.

A special example, because it appears so frequently, is the derivative with respect to time, which is often writen $\dot{y}$. Likewise the second derivative wrt time is written $\ddot{y}$.

1.4 Some essential rules

1.4.1 The product rule

Suppose we have the product of two functions, $y(x) = u(x)v(x)$, then, using eq. (1.7)

$$\begin{aligned} y(x+\delta x) =& u(x+\delta x)\, v(x+\delta x) \\ =& \left[u(x) + \frac{\mathrm{d}u}{\mathrm{d}x}\delta x + O\left(\delta x^2\right)\right]\left[v(x) + \frac{\mathrm{d}v}{\mathrm{d}x}\delta x + O\left(\delta x^2\right)\right] \\ =& u(x)v(x) + u(x)\frac{\mathrm{d}v}{\mathrm{d}x}\delta x + v(x)\frac{\mathrm{d}v}{\mathrm{d}x}\delta x + O\left(\delta x^2\right) \end{aligned} \tag{1.20}$$

Hence

$$\begin{aligned} & \lim_{\delta x\to 0}\frac{y(x+\delta x) - y(x)}{\delta x} \\ =& \lim_{\delta x\to 0}\frac{\left[u(x)v(x) + u(x)\frac{\mathrm{d}v}{\mathrm{d}x}\delta x + v(x)\frac{\mathrm{d}u}{\mathrm{d}x}\delta x + O\left(\delta x^2\right) - u(x)v(x)\right]}{\delta x} \\ =& \lim_{\delta x\to 0}\frac{\left[u(x)\frac{\mathrm{d}v}{\mathrm{d}x}\delta x + v(x)\frac{\mathrm{d}u}{\mathrm{d}x}\delta x + O\left(\delta x^2\right)\right]}{\delta x} \\ =& \lim_{\delta x\to 0}\left[u(x)\frac{\mathrm{d}v}{\mathrm{d}x} + v(x)\frac{\mathrm{d}u}{\mathrm{d}x} + O(\delta x)\right] \\ =& u(x)\frac{\mathrm{d}v}{\mathrm{d}x} + v(x)\frac{\mathrm{d}u}{\mathrm{d}x} \end{aligned} \tag{1.21}$$

Hence the product rule:

$$\frac{\mathrm{d}uv}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \tag{1.22}$$

An example is $y = x^3$. This can be written as $u = x^2$, $v = x$, so

$$\begin{aligned} \frac{\mathrm{d}x^3}{\mathrm{d}x} =& x^2\frac{\mathrm{d}x}{\mathrm{d}x} + x\frac{\mathrm{d}x^2}{\mathrm{d}x} \\ =& x^2 + x \times 2x \\ =& 3x^2 \end{aligned} \tag{1.23}$$

This suggests a general rule that if $y = x^n$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1} \tag{1.24}$$

Let us assume this is true and examine the derivative of x^{n+1}. This can be written as

$u = x^n$, $u = x$, so

$$\begin{aligned}\frac{\mathrm{d}x^{n+1}}{\mathrm{d}x} =& x^n\frac{\mathrm{d}x}{\mathrm{d}x} + x\frac{\mathrm{d}x^n}{\mathrm{d}x}\\ =& x^n + x\frac{\mathrm{d}x^n}{\mathrm{d}x}\\ =& x^n + x \times nx^{n-1}\\ =& (n+1)\,x^n \end{aligned} \tag{1.25}$$

where we have used our proposed formula in the last but one line. Note that our postulated formula is true for $n = 2$ (and $n = 1$, $n = 0$ (since $x^0 = 1$) and $n = -1$) so it is generally true.

1.4.2 The chain rule

Going back to our original example, suppose we have information about temperature, T, as a function of distance along the path, $T(s)$. We are interested in the rate of change of temperature with time. This is illustrated in Fig. 1.4. The first thing to do is chose our

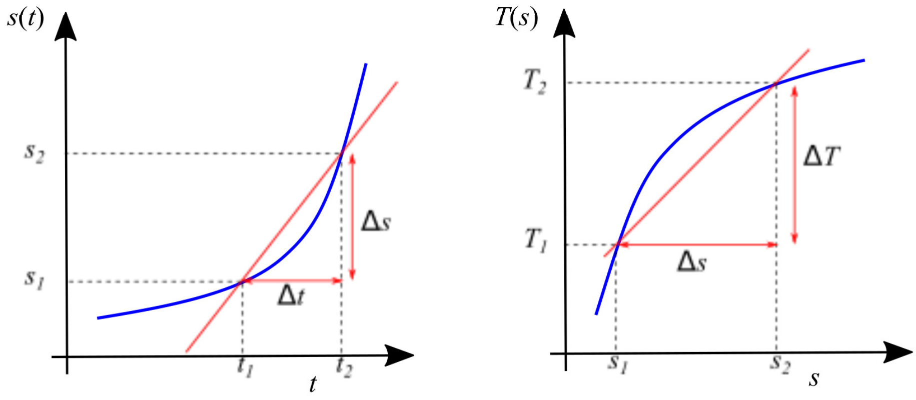

Figure 1.4: The Chain Rule: the derivative of a function of a function.

two times, t_1 and t_2. From this we can read off corresponding distances, s_1 and s_2. The rate of change of distance with time is as above,

$$\frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{s\,(t_1 + \Delta t) - s\,(t_1)}{\Delta t} \tag{1.26}$$

We can then use s_1 and s_2 to read off corresponding temperatures, T_1 and T_2. The rate of change of T with s is just

$$\frac{\Delta T}{\Delta s} = \frac{T_2 - T_2}{s_2 - s_1} = \frac{T\,(s_1 + \Delta s) - T\,(s_1)}{\Delta s} \tag{1.27}$$

We want the rate of change of T with t. This is just

$$\frac{\Delta T}{\Delta t} = \frac{T_2 - T_2}{t_2 - t_1} \tag{1.28}$$

However, we cannot write this in the form $\frac{T(t_1+\Delta t)-T(t_1)}{\Delta t}$, because T is not a function of t. We can, however, write

$$\frac{\Delta T}{\Delta t} = \frac{T_2 - T_2}{s_2 - s_1}\frac{s_2 - s_1}{t_2 - t_1} = \frac{T\left(s_1 + \Delta s\right) - T\left(s_1\right)}{\Delta s}\frac{\Delta s}{\Delta t} \tag{1.29}$$

We can now take the limit as before:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{T\left(s_1 + \Delta s\right) - T\left(s_1\right)}{\Delta s}\frac{\Delta s}{\Delta t} = \frac{\mathrm{d}T}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \tag{1.30}$$

Clearly, as $\Delta t \to 0$, so does Δs.

This leads to the *Chain Rule*:

$$\frac{\mathrm{d}}{\mathrm{d}x} f\left(g\left(x\right)\right) = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} \tag{1.31}$$

so, in our example:

$$\frac{\mathrm{d}}{\mathrm{d}t} T\left(s\left(t\right)\right) = \frac{\mathrm{d}T}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \tag{1.32}$$

1.4.3 Examples of the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x = 2\sin x \cos x \tag{1.33}$$

$f = g^2$, $g = \sin x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x\right)^{-1} = -\left[\cos x\right]^{-2}\left(-\sin x\right) = \sec x \tan x \tag{1.34}$$

$f = g^{-1}$, $g = \cos x$. Chain rule and product rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tan x =& \frac{\mathrm{d}}{\mathrm{d}x}\sin x \left(\cos x\right)^{-1}\\ =& \cos x \left(\cos x\right)^{-1} + \sin x \left[\sin x \left(\cos x\right)^{-2}\right]\\ =& 1 + \tan^2 x\\ =& \sec^2 x\end{aligned} \tag{1.35}$$

1.4.4 More notation

A notation which is highly frowned on by mathematicians is, nevertheless, often used. It is useful shorthand *if used carefully*! Effectively we multiply by the infinitesimal increment in our dependent variable. Thus, for example, if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \tag{1.36}$$

then we write

$$\mathrm{d}y = 2x\mathrm{d}x \tag{1.37}$$

This is very poor notation as it really has no meaning; in practice we are effectively stating the chain rule by saying that to obtain a derivative with respect to *any* variable, say ξ, we just divide both sides by $\mathrm{d}\xi$ thus:

$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = 2x\frac{\mathrm{d}x}{\mathrm{d}\xi} \tag{1.38}$$

or, combined with the notation discussed above

$$\frac{\mathrm{d}x^2}{\mathrm{d}\xi} = 2x\frac{\mathrm{d}x}{\mathrm{d}\xi} \tag{1.39}$$

This is also useful when we come to integration.

We use this notation a lot in thermodynamics. For example, if q is the heat absorbed by unit mass of a fluid, we write:

$$\mathrm{d}q = C_v\mathrm{d}T + p\mathrm{d}\alpha \tag{1.40}$$

where C_v is the specific heat capacity at constant volume, T is the temperature, p the pressure and α the specific volume. See the total derivative, Sec. 1.6 below, for further discussion.

1.5 Partial derivatives

We often deal with functions of more than one variable. In meteorology, for example, the temperature, T may be a function of three spatial dimensions, and also time, i.e. $T = T(x, y, z, t)$. The partial derivative is simply the derivative with respect to one variable. However, to distinguish this from the *total derivative*, below, it has a special notation using a lower case d in a special font, ∂ (\partial in LaTeX). Thus:

$$\frac{\partial T}{\partial x} = \lim_{\delta x \to 0} \frac{T(x + \delta x, y, z, t) - T(x, y, z, t)}{\delta x} \tag{1.41}$$

$$\frac{\partial T}{\partial y} = \lim_{\delta y \to 0} \frac{T(x, y + \delta y, z, t) - T(x, y, z, t)}{\delta y} \tag{1.42}$$

etc..

The important point is that all the other variables are held at a fixed value. The rules above are equally valid so long as care is taken to all the other variables fixed (i.e. treat them as constants). For example, suppose $T = \sin x \cos y$, then

$$\frac{\partial T}{\partial x} = \cos x \cos y \tag{1.43}$$

$$\frac{\partial T}{\partial y} = -\sin x \sin y \tag{1.44}$$

It is often helpful to remind ourselves what variables are being held constant. This is particularly true in meteorology, where we often use the same variable to denote a field representign a physical quantity (e.g. T) irrespective of what coordinate system we are using. Thus, we might write $T(t, x, y, z)$ and $T(t, \lambda, \phi, p)$, the former being in terms of 3D Cartesian coordinates, the latter in terms of longitude, latitude and hydrostatic pressure. Both represent the same field, though they may be very different functions expressed in the two coordinate systems. In this case, it is often useful to write partial derivatives as follows:

$$\left.\frac{\partial T}{\partial x}\right|_{t,y,z} \tag{1.45}$$

to make it clear that the derivative is on surfaces of constant y and z and at constant t.

1.6 The total derivative

Suppose we have some function of many variables. Suppose, for example, we have terrain height h as a function of horizontal position x and y. We may have some path defined, say in terms of positions x and y as a function of t (say time). Then the *Total Derivative* is defined as

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{h(x(t+\delta t), y(t+\delta t)) - h(x(t), y(t))}{\delta t} \tag{1.46}$$

The numerator of the ratio we are taking the limit of can be expanded using partial derivatives:

$$\begin{aligned} h(x(t+\delta t), y(t+\delta t)) - h(x(t), y(t)) =& h(x(t), y(t+\delta t)) + \frac{\partial h}{\partial x}\delta x + O\left(\delta x^2\right) - h(x(t), y(t)) \\ =& h(x(t), y(t+\delta t)) + \frac{\partial h}{\partial x}\delta x + O\left(\delta x^2\right) - h(x(t), y(t)) \\ =& h(x(t), y(t)) + \frac{\partial h}{\partial x}\delta x + O\left(\delta x^2\right) \\ &+ \frac{\partial h}{\partial y}\delta y + O\left(\delta y^2\right) - h(x(t), y(t)) \\ =& \frac{\partial h}{\partial x}\delta x + O\left(\delta x^2\right) + \frac{\partial h}{\partial y}\delta y + O\left(\delta y^2\right) \end{aligned} \tag{1.47}$$

where $\delta x = x(t+\delta t) - x(t)$ and $\delta y = y(t+\delta t) - y(t)$.

Thus

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{\frac{\partial h}{\partial x}\delta x + O(\delta x^2) + \frac{\partial h}{\partial y}\delta y + O(\delta y^2)}{\delta t} = \frac{\partial h}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial h}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{1.48}$$

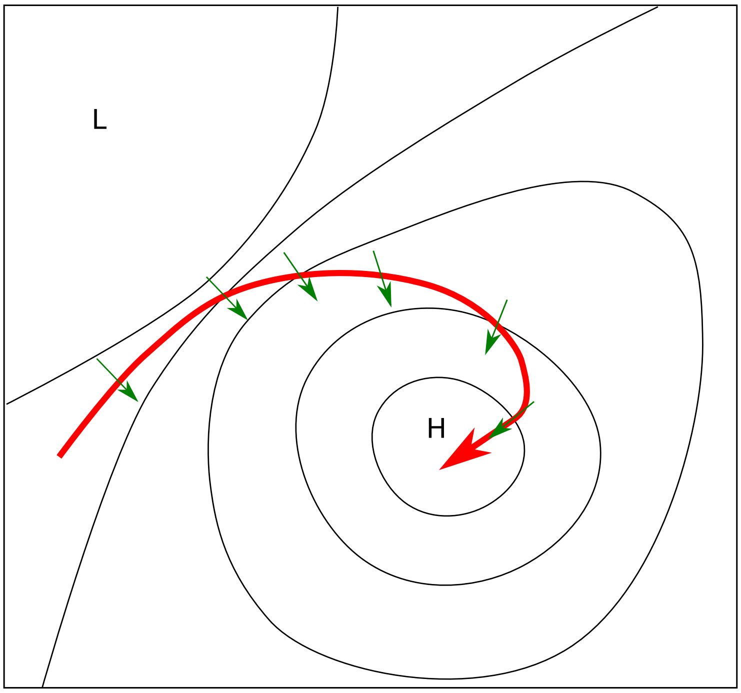

Figure 1.5: Contours of height, h, equally spaced in h (black). The rate of change of h following a path (red) depends upon the component of the slope, the direction of which is shown by green arrows, *along the path*. Initially the slope has no uphill component, so the rate of change of height along the path is zero. Towards the end, the path is aligned with the slope so the rate of change of height along the path is large.

1.6.1 The gradient operator

The above suggests that, in Cartesian coordinates, we can separate the total derivative into two parts. One is a vector which is a function of the field h only, $\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)$. The second is a function of the path, $\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right)$, which is just the vector velocity, $\mathbf{v}$, along the path.

The first of these is so important, we define the operator **grad**. In Cartesian coordinates, **grad**h is the vector

$$\mathbf{grad}h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) \equiv \nabla h \tag{1.49}$$

where we have defined the operator $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ (in two dimensions - equivalents may be defined in more dimensions).

Note that **grad** means the vector gradient in any coordinate system. It only equals the ∇ operator in Cartesian coordinates, and my look very different in another coordinate system.

With these definitions, using the vector dot product:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \mathbf{grad}h \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{grad}h = \nabla h \cdot \mathbf{v} = \mathbf{v} \cdot \nabla h \tag{1.50}$$

Furthermore, if we recognise that the velocity is just the rate of change of distance, with a vector increment $\mathrm{d}\mathbf{l} = (\mathrm{d}x, \mathrm{d}y)$, then an infinitesimal change in height *along the path* is given by

$$\mathrm{d}h = \mathbf{grad}h \cdot \mathrm{d}\mathbf{l} = \nabla h \cdot \mathrm{d}\mathbf{l} \tag{1.51}$$

This is illustrated in Fig. 1.5.

1.7 The Lagrangian or Material derivative

In fluid mechanics, the laws of physics are most easily expressed and understood in terms of the properties of specific (infinitesimally small) elements or parcels of fluid. If such a parcel has a property, such as temperature, T, we can define a derivative of that temperature with respect to, say, time T. To make it clear that this derivative is *following a fluid parcel*, we give it a special notation, but the definition is precisely as usual:

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \lim_{\delta t \to 0} \frac{T_{parcel}\left(t + \delta t\right) - T_{parcel}\left(t\right)}{\delta t} \tag{1.52}$$

If we label each parcel somehow (e.g. with it's position at some reference time, t_0, i.e. $\mathbf{x}_0$), then knowing the position of each parcel, $\mathbf{x}_{parcel}\left(t, \mathbf{x}_0\right)$ tells us the entire fluid flow. This is known as a Lagrangian formulation of fluid mechanics, and the derivative above is known as the *Lagrangian* or *material* derivative. Using this makes writing down Newton's second law very simple:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{1}{\rho}\mathbf{F} \tag{1.53}$$

where ρ is the density of and $\mathbf{F}$ is the nett force on the parcel and

$$\mathbf{u} \equiv \frac{\mathrm{D}}{\mathrm{D}t}\mathbf{x}_{parcel} \tag{1.54}$$

However, in practice, this approach his some difficulties - for example deriving the pressure field. Instead, we use fields of, say, temperature or velocity as functions of spatial coordinates, i.e. $T\left(t, x, y, z\right)$ or $T\left(t, \mathbf{x}\right)$, $\mathbf{u}\left(t, x, y, z\right)$ or $\mathbf{u}\left(t, \mathbf{x}\right)$. We call this an *Eulerian* representation.

The material derivative is really just the total derivative defined above in the special case that the path followed is that of the fluid parcels. It can be calculated in terms of partial derivatives of the Eulerian fields following the same reasoning, bearing in mind that the path is a function of t and the fields may also be varying as a function of t. Thus

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\mathrm{D}x}{\mathrm{D}t} + \frac{\partial T}{\partial y}\frac{\mathrm{D}y}{\mathrm{D}t} + \frac{\partial T}{\partial x}\frac{\mathrm{D}z}{\mathrm{D}t} \tag{1.55}$$

We note that $\frac{\mathrm{D}x}{\mathrm{D}t}$ is just the x component of the fluid velocity, u, and similarly for the other components, so the above can be written:

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \tag{1.56}$$

Note that the two 'T' variables are strictly not the same functions. On the left T is a function of parcel label and time. On the right, T is a function of fixed space location and time. We use the same letter to signify that, as physical quantities, the two must be equal where the parcel position coincides with a given point in space, i.e. $T_{parcel}\left(t, \mathbf{x}_0\right) = T\left(t, \mathbf{x}_{parcel}\left(t, \mathbf{x}_0\right)\right)$