Chapter 3

Complex Numbers

3.1 The origin of complex numbers

We are all familiar with the idea that some quadratic equations have no (real) solution. The general solution to $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.1}$$

this is arrived at by the process of 'completing the square', i.e. noting that $\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 = ax^2 + bx + \frac{b^2}{4a}$ so if we add and subtract $\frac{b^2}{4a}$ to the original equation, it can be written $\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a} = 0$. Thus $\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right) = \pm\sqrt{\frac{b^2}{4a} - c}$ etc.. Hence, if $b^2 < 4ac$ there is no real solution. Introducing $i = \sqrt{-1}$ then makes the general solution always valid by providing a solution even when no 'real' solution exists.

However, this is not the origin of complex numbers - mathematicians had known for centuries that some quadratic equations had no solution, and did not see that a a problem - they know why (the parabola represented by the equation does not intersect the x axis), so merely saw the 'discriminant' as a useful rule to tell us there is no solution.

The problem arises because a similar formula exists for cubic equations. Mathematicians in the 16th century knew that this method required taking (cube) roots of negative numbers even when real solutions were known to exist. It was shown that, in such cases, if one merely keeps track of were one needs the root of -1, these terms eventually cancel if the method is applied to such problems. This lead to the idea of defining imaginary and complex numbers.

3.2 The algebra of complex numbers

A complex number is generally written $z = x+iy$ where x and y are both real numbers and $i^2 = -1$. x is the 'real part' and iy, or just y, taking the factor i as read, the 'imaginary part'. Note that electrical engineers usually use j instead of i, as i is used for electric current.

In any mathematical operation, we can always separate the result into the imaginary and real parts by finding the terms including i (after simplifying products like $i \times i$) and those not including i.

Addition (and hence subtraction) is simple: if $z_1 = a + ib$, $z_2 = c + id$ then

$$z_1 + z_2 = a + ib + c + id = (a + c) + i\,(b + d) \tag{3.2}$$

Multiplication just requires carefully keeping track of terms and re-organising the result:

$$\begin{aligned} z_1 \times z_2 =& (a + ib) \times (c + id) \\ =& ac + aid + ibc + ibid \\ =& (ac - bd) + i\,(ad + bc) \end{aligned} \tag{3.3}$$

Note that if $a = d$ and $c = -b$ we obtain $z_1 z_2 = a^2 + b^2$. This is so important, we give the complex number with the opposite sign imaginary part the special name *complex conjugate* and denote the complex conjugate of z as z^*. This, if $z = x + iy$, $z^* = x - iy$ and $zz^* = x^2 + y^2$. The latter is defined as the *magnitude* of z.

Division is less obvious - it is best always to think in terms of multiplication. Consider $\frac{z_2}{z_1}$

$$\frac{z_2}{z_1} = \frac{(c + id)}{(a + ib)} \tag{3.4}$$

$$= \frac{(c + id)\,(a - ib)}{(a + ib)\,(a - ib)} \tag{3.5}$$

$$= \frac{(ac + bd) + i\,(ad - bc)}{a^2 + b^2} \tag{3.6}$$

3.2.1 Relationship with complex numbers and the Argand diagram

We showed in the previous chapter that

$$\exp(ix) = \cos x + i \sin x \tag{3.7}$$

If we plot a point in a 2D Cartesian coordinate system at (x, y) then we note (Fig. 3.1) that the vector joining the origin to that point makes an angle θ with the x axis and has length r, with:

$$x = r \cos\theta \tag{3.8}$$

$$y = r \sin\theta \tag{3.9}$$

so

$$r = \sqrt{(x^2 + y^2)} \tag{3.10}$$

$$\theta = \arctan \frac{y}{x} \tag{3.11}$$

Suppose we identify the point (x, y) with the complex number $z = x + iy$, then from eq. (3.7), we can also write

$$z = r \exp(i\theta) \equiv re^{i\theta}. \tag{3.12}$$

This is an extremely useful identity, as the next chapter will show. Note that the whole *complex plane* is spanned by $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$ or $\theta \in [-\pi, \pi)$. Using the expo-

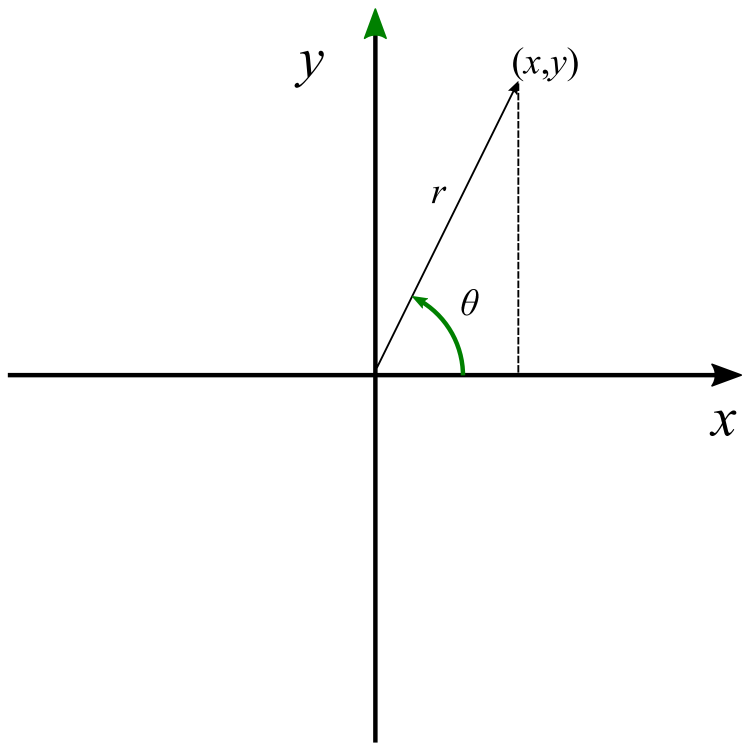

Figure 3.1: The Argand diagram.

nential notation, multiplication and division of complex numbers becomes much simpler. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 \times z_2 = r_1 e_2^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)} \tag{3.13}$$

$$\frac{z_2}{z_1} = \frac{r_2 e^{i\theta_2}}{r_1 e^{i\theta_1}} = \frac{r_2}{r_1} e^{i(\theta_2 - \theta_1)} \tag{3.14}$$

Note that $z^* = re^{-i\theta}$ in general, so $zz^* = re^{i\theta} re^{-i\theta} = r^2$ as expected.

If we use complex numbers with $r = 1$, i.e. on the *unit circle*, then comparing eq. (3.13) with eq. (3.3) we recover two trigonometric identities for very little work:

$$\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) = (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2) \tag{3.15}$$

This illustrates that we can make our lives much easier by freely converting between the $re^{-i\theta}$ and the $r\cos\theta + ir\sin\theta$ notation - the former is valuable for multiplication and division (and, as we shall see, powers and roots), the latter for addition and subtraction.

3.2.2 Powers and roots of complex numbers

Powers are simply expressed using $z = re^{-i\theta}$. Clearly

$$z^\alpha = r^\alpha e^{-i\theta\alpha} \tag{3.16}$$

Roots are similarly simple. Suppose $z_n = ce^{i\beta}$ is the nth root of $z = re^{i\theta}$. Then

$$z_n^n = c^n e^{in\beta} \equiv re^{i\theta} \tag{3.17}$$

Thus $c = r^{\frac{1}{n}}$ and $\beta = \theta/n$. However, we can also write $z = re^{i(2m\pi+\theta)}$ for any integer m, so we have a set of roots $c = r^{\frac{1}{n}}$ and $\beta = (2m\pi + \theta)/n$. These roots are unique for all values of m from 0 to $n-1$.

This leads to a number of useful results:

The nth roots of -1 are $e^{i(2m+1)\pi/n}$ since $-1 = e^{i(2m+1)\pi}$. -1 has 2 square roots, $i = e^{i\pi/2}$ and $-i = e^{i3\pi/2}$.

The nth roots of i are $e^{i(2m+1/2)\pi/n}$ since $i = e^{i(2m+1/2)\pi}$. i has 2 square roots, $e^{i\pi/4} = 1/\sqrt{2} + i/\sqrt{2}$ and $e^{i5\pi/4} = -1/\sqrt{2} - i/\sqrt{2}$.

The nth roots of 1 are $e^{i2m\pi/n}$ since $1 = e^{i2m\pi}$. 1 has 2 square roots, $e^{i0} = 1$ and $e^{i\pi} = -1$. 1 has 3 cube roots, 1, $e^{i2\pi/3} = -1/2 + i\sqrt{3}/2$ and $e^{i4\pi/3} = -1/2 - i\sqrt{3}/2$.

Note that when you look at Discrete Fourier Series, the series will be a sum over the n roots of 1.