

### The Penman equation

For agricultural or hydrological applications it is essential to know surface parameters such as surface evaporation or heat fluxes. However, these parameters are quite difficult to measure directly. For humid surfaces, the Penman equation<sup>1</sup> gives a useful approximation for these fluxes as well as surface properties in terms of quantities which can be measured by standard meteorological instruments.

Assume the surface air temperature is  $T_0$ . We also assume that the near-surface air is saturated, so that its local vapour pressure is  $e_s(T_0)$ . We can measure the temperature  $T$  and humidity  $q$  at an elevation  $H$ . We next assume that the heat and humidity fluxes, see Section 3.7, can be approximated using bulk-aerodynamic approximations. For example, we take the sensible heat flux  $F_h$  to be a Fickian vertical diffusion,

$$F_h = \bar{\rho} c_p \overline{w'T'} \approx -\rho c_p K_h \partial T / \partial z \approx \rho c_p K_h (T_0 - T) / H, \quad (1)$$

where  $K_h$  is the effective eddy diffusion coefficient for heat and where in the last step the partial derivative has been replaced by a finite difference over depth  $H$ . A similar equation is valid for the latent heat flux and we arrive at the following set of approximate equations for the sensible and latent heat fluxes:

$$F_h = \rho c_p C_h (T_0 - T), \quad (2)$$

$$F_l = \rho L C_l (q_0 - q), \quad (3)$$

with  $q_0$  the (unknown) specific humidity at the surface, and the bulk transfer coefficients  $C_h = K_h/H$  and  $C_l = K_l/H$ .<sup>2</sup> Note that in the above definition,  $C_h$  and  $C_l$  have units of velocity. Indeed, if the heat and moisture transfer occurs through turbulence, it makes sense that these coefficients are a measure of the turbulent velocities. This is formalized in the so-called *mixing length hypothesis*, where it is taken that  $C_h$  and  $C_l$  are proportional to a local velocity scale, normally taken to be the absolute value of the measured horizontal velocity. In the boundary layer literature, bulk transfer coefficients are defined as the nondimensional coefficients which multiply the local velocity to get the above transfer coefficients.

Because the vertical fluxes are generally due to turbulent diffusion rather than molecular diffusion, it is not a bad approximation to take  $C_l = C_h$ .

<sup>1</sup>Penman, H. L., 1948: Natural evaporation from open water, bare soil and grass. *Proceedings of the Royal Society of London, Series A*, **193**, 120–145; Monteith, J., and M. Unsworth, 1990: *Principles of Environmental Physics*, Butterworth–Heinemann, 291pp.

<sup>2</sup>In the agro-meteorological literature these transfer coefficients are often written in terms of their inverse, the resistance  $r$ , e.g.,  $C_h = 1/r_h$ . This convention is inspired by electronics analogs: media with different transfer coefficients are combined in the same way as parallel or serial resistors.

We next rewrite  $F_l$  in terms of the vapour pressure. In order to do this we note that for small specific humidities we have

$$q_0 - q = \frac{R_d}{pR_v} (e_s(T_0) - e) = \frac{R_d}{pR_v} (e_s(T) - e + \Delta(T - T_0)), \quad (4)$$

where in the last step we linearized the dependence of the saturation vapour pressure on temperature and used the Clausius–Clapeyron equation to define

$$\Delta = \frac{de_s}{dT} = \frac{Le_s(T)}{R_v T^2}. \quad (5)$$

We assume  $\Delta$  to be constant in the surface layer, that is, the surface temperature and the air temperature at elevation  $H$  are not too far apart. The value of  $\Delta$  itself varies considerably with temperature, from about  $44 \text{ Pa K}^{-1}$  at  $T = 0^\circ\text{C}$  to  $144 \text{ Pa K}^{-1}$  at  $T = 20^\circ\text{C}$ .

We next assume that the energy balance at the surface is steady. This means that the latent and sensible heat fluxes are completely balanced by the net downward radiation, indicated as  $R_n$ ,

$$R_n = F_h + F_l. \quad (6)$$

This equation implicitly assumes that the net heat flux into the ground can be ignored. In principle this ground heat flux could be included in the budget although in practice it is difficult to measure. All we need to do is replace everywhere  $R_n$  by  $R_n - G$ , where  $G$  is the heat flux into the ground. Generally the net ground heat flux is small when averaged over time scales longer than a day.

We now have prepared all the ingredients to write down the Penman equation for the latent heat flux. Substituting Eq. 4 in Eq. 3, we have

$$F_l = \frac{\rho L C_l R_d}{p R_v} (e_s(T) - e + \Delta(T - T_0)), \quad (7)$$

$$= \frac{\rho L C_l R_d}{p R_v} \left( e_s(T) - e + \Delta \frac{R_n - F_l}{\rho c_p C_h} \right), \quad (8)$$

where we used  $F_h = R_n - F_l$  to rewrite  $T - T_0$  in the first equation. This equation can be rearranged to find the *Penman equation*

$$\blacktriangleright \quad F_l = \frac{\Delta R_n + \rho c_p C_h (e_s(T) - e)}{\Delta + \gamma^*}, \quad (9)$$

where we have introduced the effective psychrometric constant for vapour pressure,

$$\gamma^* = \frac{c_p p R_v C_h}{L R_d C_l}. \quad (10)$$

A typical value for  $\gamma^*$  is  $65 \text{ Pa K}^{-1}$  where we assume that  $C_l = C_h$ . The values of  $\gamma^*$  and  $\Delta$  are the same at a temperature of about  $6^\circ\text{C}$ .

We have followed the traditional notation for the Penman equation, but it is perhaps more insightful to rewrite it in the form

$$F_l = \frac{\Delta R_n + \gamma^* F_{lm}}{\Delta + \gamma^*} \quad (11)$$

where  $F_{lm}$  is the latent heat flux calculated using the actual measured temperature instead of the surface temperature,

$$F_{lm} = \rho L C_l (q_s(T) - q). \quad (12)$$

In this form of the Penman equation, it is clear that at high temperatures (large  $\Delta$ ), the humidity deficit plays a smaller role in setting the latent heat flux.

The Penman equation can be used to derive the other unknowns in the heat budget. We find for the sensible heat flux

$$F_h = \frac{\gamma^* R_n - \rho c_p C_h (e_s(T) - e)}{\Delta + \gamma^*} = \frac{\gamma^* R_n - \gamma^* F_{lm}}{\Delta + \gamma^*}. \quad (13)$$

The Bowen ratio is

$$\beta = \frac{F_h}{F_l} = \frac{\gamma^* R_n - \gamma^* F_{lm}}{\Delta R_n + \gamma^* F_{lm}}. \quad (14)$$

The evaporative fraction EF is

$$\text{EF} = \frac{F_l}{F_h + F_l} = \frac{\Delta + \gamma^* F_{lm}/R_n}{\Delta + \gamma^*}. \quad (15)$$

These expressions become particularly simple when  $F_{lm}$  is substantially smaller than  $R_n$ . This can be the case for low wind speeds, (because  $C_l$  is typically proportional to the wind speed at elevation  $H$ ), high relative humidity (because  $e_s(T) - e = e_s(T) (1 - \text{RH})$ ) or high net radiation. In this situation the Bowen ratio  $\beta$  approaches  $\gamma^*/\Delta$  which varies from 1.5 at  $T = 0^\circ\text{C}$  to 0.5 at  $T = 20^\circ\text{C}$ .

The surface air temperature can also be derived from the Penman equation and the related expression for  $F_h$ , namely

$$T_0 = T - \frac{e_s(T) - e}{\Delta + \gamma^*} + \frac{\gamma^*}{\Delta + \gamma^*} \frac{R_n}{\rho c_p C_h} = T + \frac{\gamma^*}{\Delta + \gamma^*} \frac{R_n - F_{lm}}{\rho c_p C_h}. \quad (16)$$

It is straightforward to modify the above arguments for fixed relative humidity  $\text{RH}_0$  at the surface. The only change is the relation between  $q_0$  in Eq. 3 and the saturated vapour pressure at the surface. The Penman equation and those equations following from it are modified by replacing  $e_s$  by  $\text{RH}_0 e_s$ , and  $\Delta$  by  $\text{RH}_0 \Delta$ .