The Propagation Mechanism of a Vortex on the β Plane

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ABSTRACT

The propagation velocity and propagation mechanism for vortices on a β plane are determined for a reduced-gravity model by integrating the momentum equations over the β plane. Isolated vortices, vortices in a background current, and initial vortex propagation from rest are studied. The propagation mechanism for isolated anticyclones as well as cyclones, which has been lacking up to now, is presented. It is shown that, to first order, the vortex moves to generate a Coriolis force on the mass anomaly of the vortex to compensate for the force on the vortex due to the variation of the Coriolis parameter. Only the mass *anomaly* of the vortex is of importance, because the Coriolis force due to the motion of the bulk of the layer moving with the vortex is almost fully compensated by the Coriolis force on the motion of the exterior flow. Because the mass anomaly of a cyclone is negative the force and acceleration have opposite sign. The role of dipolar structures in steadily moving vortices is discussed, and it is shown that their overall structure is fixed by the steady westward motion of the mass anomaly. Furthermore, it is shown that reduced-gravity vortices are not advected with a background flow. The reason for this behavior is that the background flow changes the ambient vorticity gradient such that the vortex obtains an extra self-propagation term that exactly cancels the advection by the background flow. Last, it is shown that a vortex initially at rest will accelerate equatorward first, after which a westward motion is generated. This result is independent of the sign of the vortex.

1. Introduction

Several expressions have been derived for the propagation speed of oceanic monopoles in the literature. Even multilayer vortices with noncircular shapes have been addressed. Flierl et al. (1983) and Dewar (1988) mention that the propagation of an anticyclone in steady motion is needed to compensate for the net Coriolis force on the swirling motion in the vortex. As is shown below, their explanation is correct for lenses, but a subtle point has to be added when the active layer depth does not vanish at infinity. Nof (1983) considers inner and outer regions for vortices, and finds a pressure force exerted on the vortex by the exterior flow. This pressure force is coined the planetary lift, in accordance with the lift force of a solid body with nonzero circulation. The origin of this lift force is not clear. Larichev (1984) shows that in a quasigeostrophic barotropic ocean, expressions for the motion of the center of mass of the vortex obtained from the momentum equation can also be derived directly from the quasigeostrophic potential vorticity equation. Cushman-Roisin et al. (1990) give a more complete explanation for the results they derive, but their arguments are incorrect, as shown below. Nycander (1996) shows that the motion of an anticyclone of constant shape, described as a spinning disk, can be understood in an inertial frame of reference by the conservation of angular momentum as precession, but he mentions that an explanation for the westward motion of cyclones is still lacking. In addition, McDonald (1998) shows that inertial oscillations of a rotating disk can be understood in terms of nutation. The propagation of an anticyclonic in a reduced-gravity context on a sphere is presented by Van der Toorn (1997) by considering the total angular momentum of the rotating earth with a spinning vortex. He shows that a mass anomaly has to make a precession motion around the rotation axis of the earth, related to the torque from the gravity.

In this paper we show that the motion of both anticyclones and cyclones in a coordinate system attached

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to the rotating earth is fully understandable as well. This might not come as surprise given the work of Van der Toorn (1997), but the derivation is much simpler and closer to the description used in physical oceanography. The rationale for concentrating on a reduced-gravity description is that we consider surface intensified vortices, so that the body of water that moves with the vortex, which is most important from a thermohaline point of view, is mainly concentrated in the upper layers. The idea used here is to study the motion of the center of mass of the vortex by integrating the momentum equations over the entire domain. This leads to much simpler and easier to understand expressions for the vortex propagation than studies that also try to unravel the details of the vortex structure (e.g., Flierl 1987; Flierl et al. 1980; Sutyrin et al. 1994; Sutyrin and Dewar 1992; Reznik and Dewar 1994; Benilov 1996; Stern and Radko 1998; Reznik and Grimshaw 2001). Also, it adds new insight because it stresses the role of the fluid that is not propagating with the vortex. In several papers the motion of a vortex is explained in terms of secondary dipolar circulations (e.g., Sutyrin and Dewar 1992; Reznik and Dewar 1994; Benilov 1996; Stern and Radko 1998; Reznik and Grimshaw 2001). The idea is that the propagating vortex pushes surrounding water north- and southward, creating a dipolar structure that advects the vortex westward. However, the advection of the dipolar mass anomalies by the monopole is neglected, while it can be shown to be of similar magnitude to the advection of the monopole by the dipole (see section 2b). Furthermore, Nycander and Sutyrin (1992) have shown that these dipolar fields can have such orientation that they tend to decrease the westward motion of the vortex. An explanation of vortex motion by secondary dipolar circulations is thus incomplete. In this paper the influence of dipolar fields on vortex propagation is investigated, and it is explained why dipoles only enter the expression for the propagation speed via their mass anomaly, and not by advective terms.

By examining the influence of a geostrophic current on vortex propagation in a reduced-gravity model, it becomes clear that vortices are not just advected by such a current. On the contrary, to first order the background current has no influence on the propagation speed at all. So advection of Gulf Stream vortices by the gyre circulation, or of Agulhas rings by the Benguela Current has to be reconsidered.

Last, we investigate the initial motion of a vortex on in the β plane is investigated. In a barotropic quasi-geostrophic model Adem (1956) has shown that a cyclone moves westward first, after which a poleward motion develops. By considering the motion of the center of mass of the vortex, it is shown here that all vortices first move equatorward, after which a westward velocity component develops. This result is true for vortices of both signs. The discrepancy between these two results is resolved, and it is shown that the vortex by Adem (1956) is not at rest initially. The same initial motion of the vortex is present in articles by Reznik and Dewar (1994), Benilov (2000, when he discusses the compensated vortex), and Reznik and Grimshaw (2001). The deeper reason for this nonzero initial motion is that the initial vortex is assumed to have a radial pressure distribution, leading to asymmetric initial water parcel velocities, as will be explained in section 4.

In the next section a simple derivation of the motion of a vortex on the β plane is given. Previous explanations for vortex motion are critically discussed and the counterintuitive role of dipolar fields in the vortex is emphasized. Then the influence of a background flow is determined, followed by an investigation of the initial motion of a vortex. The paper is completed by a concluding section and an appendix.

2. The propagation of a vortex on the β plane

The momentum equations for the reduced-gravity model on the β plane read

$$\frac{du}{dt} - fv = -g'\eta_x \quad \text{and} \tag{1}$$

$$\frac{dv}{dt} + fu = -g'\eta_y, \tag{2}$$

in which u, v, g', and f are conventional, and η is the interface displacement measured positive downward. Mass conservation is expressed by volume conservation as

$$\frac{dh}{dt} + h(u_x + v_y) = 0, \tag{3}$$

where the layer thickness $h = H + \xi + \eta$, ξ is the surface elevation, and H is the depth of the undisturbed first layer. From now on, we neglect the surface elevation ξ relative to η.

We assume that the center of mass of the first layer is a good description of the position of the vortex. The center of mass of the active layer is given by

$$X = \frac{\int x\eta\, dA}{\int \eta\, dA} = \frac{1}{V}\int x\eta\, dA \quad \text{and} \tag{4}$$

$$Y = \frac{\int y\eta\, dA}{\int \eta\, dA} = \frac{1}{V}\int y\eta\, dA, \tag{5}$$

in which V is the anomalous volume of the first layer. The integration is performed over the entire β plane. Although the emphasis of this paper is on monopolar vortices, the results are in fact more general as will become clear from the derivations given below.

The center-of-mass translation is not a good description of the motion of a vortex if the vortex emits a relatively large amount of mass (e.g., by filamentation). One might argue that the maximum surface elevation of the vortex gives a better measure of the position of the vortex. However, Nycander and Sutyrin (1992) and Cushman-Roisin et al. (1990) mention that in their numerical experiments the center of mass and the maximum surface elevation of the vortex followed the same trajectory. Furthermore, for a steadily translating vortex the center of mass and the maximum surface elevation have to move with the same speed. In the appendix it is shown that the center of potential vorticity follows the center of mass to dominant order. Furthermore, numerical experiments are discussed in that appendix, which show that the difference between the center of mass and the maximum interface elevation is very small.

When the center of mass is accepted as the position of the vortex, the translation velocity of the vortex is the time derivative of the center of mass:

$$V\frac{dX}{dt} = \int x\eta_t\, dA = -\int x[(hu)_x + (hv)_y]\, dA = \int hu\, dA \quad \text{and} \tag{6}$$

$$V\frac{dY}{dt} = \int y\eta_t\, dA = -\int y[(hu)_x + (hv)_y]\, dA = \int hv\, dA. \tag{7}$$

Here we used that V is time independent in a volume-conserving system. The idea now is to study the area-integrated force balance on the vortex. Because the Coriolis force on a moving vortex is linearly related to the velocity of the vortex, an expression for the latter is obtained. So, to proceed we differentiate the expression for the velocity of the vortex to the following time:

$$V\frac{d^2X}{dt^2} = \int (hu)_t\, dA = -\int [(hu^2)_x + (huv)_y - fhv + g'h\eta_x]\, dA \quad \text{and} \tag{8}$$

$$V\frac{d^2Y}{dt^2} = \int (hv)_t\, dA = -\int [(huv)_x + (hv^2)_y + fhu + g'h\eta_y]\, dA, \tag{9}$$

from which we find

$$V\frac{d^2X}{dt^2} = \int fhv\, dA \quad \text{and} \tag{10}$$

$$V\frac{d^2Y}{dt^2} = -\int fhu\, dA. \tag{11}$$

These equations show that zonal and meridional accelerations of the center of mass of the vortex arise when the net Coriolis force on the vortex is nonzero.

The Coriolis force at the right-hand side of these expressions still contain the propagation speed of the vortex. When we bring that part to the left-hand side we obtain, also using also $f = f_0 + \beta y$, the following:

$$V\frac{d^2X}{dt^2} = V\frac{dY}{dt}(f_0 + \beta Y) + \int \left(hv - \eta\frac{dY}{dt}\right)\beta y\, dA \quad \text{and} \tag{12}$$

$$V\frac{d^2Y}{dt^2} = -V\frac{dX}{dt}(f_0 + \beta Y) - \int \left(hu - \eta\frac{dX}{dt}\right)\beta y\, dA. \tag{13}$$

These are the basic equations for vortex propagation on the β plane. To obtain an order of magnitude of the flow field we assume the mass flow apart from the propagation of the vortex, to be in geostrophic balance. Furthermore, by a suitable choice of the initial conditions (and consistent with the β-plane approximation) we have $\beta Y \ll f_0$. This leads to

$$\frac{d^2X}{dt^2} = f_0\frac{dY}{dt} \quad \text{and} \tag{14}$$

$$\frac{d^2Y}{dt^2} = -f_0\frac{dX}{dt} - f_0\beta R_d^2\left(1 + \frac{1}{V}\int \frac{1}{2}\frac{\eta^2}{H}\, dA\right), \tag{15}$$

in which $R_d = (g'H)^{1/2}/f_0$ is the Rossby radius of deformation. These are the equations we will use in the following sections.

Another approximation would be to assume cyclogeostrophic balance for the swirling motion:

$$f_0 U = g'\eta_r - \frac{U^2}{r}, \tag{16}$$

in which r is the radial coordinate originating in the center of the vortex, and U is the swirl velocity, which is positive for cyclonic motion and negative for anticyclonic motion. We then find

$$\int \left(h\upsilon - \eta \frac{dY}{dt} \right) \beta y \, dA = \beta \int h(U\cos\theta)\, r\sin\theta r\, dr\, d\theta = 0 \quad \text{and} \tag{17}$$

$$-\int \left(hu - \eta \frac{dX}{dt} \right) \beta y \, dA = -\beta \int h(-U\sin\theta)\, r\sin\theta r\, dr\, d\theta = \beta \int hUr\sin^2\theta r\, dr\, d\theta$$
$$= \beta \int hr \left(\frac{g'\eta_r}{f_0} - \frac{U^2}{rf_0} \right) \sin^2\theta r\, dr\, d\theta. \tag{18}$$

Partial integration of the latter equation gives in (12) and (13), again assuming $\beta Y \ll f_0$,

$$\frac{d^2X}{dt^2} = f_0 \frac{dY}{dt} \quad \text{and} \tag{19}$$

$$\frac{d^2Y}{dt^2} = -\frac{dX}{dt} f_0 - f_0 \beta R_d^2 \times \left[1 + \frac{1}{Vg'H} \int \left(\frac{1}{2} g'\eta^2 + \frac{1}{2} hU^2 \right) dA \right], \tag{20}$$

in which we recognize the sum of kinetic and available potential energy in the latter two terms. We will come back to this later.

a. Small vortex accelerations

To understand the contribution of the different terms we consider vortices with negligible acceleration. From a formal-scale analysis one can argue that the acceleration terms are smaller than the other terms (Cushman-Roisin et al., 1990), leading to

$$V \frac{dX}{dt} (f_0 + \beta Y) = -\int \left(hu - \eta \frac{dX}{dt} \right) \beta y \, dA \quad \text{and} \tag{21}$$

$$V \frac{dY}{dt} (f_0 + \beta Y) = -\int \left(h\upsilon - \eta \frac{dY}{dt} \right) \beta y \, dA. \tag{22}$$

For a steadily translating vortex these equations are exact. Note that because the motion is steady any reservations one can have with taking the center-of-mass velocity as the propagation speed of the vortex vanishes because all measures for this speed, like the interface-anomaly maximum, or the relative vorticity maximum, are the same. The equations tell us that the Coriolis force on the *mass anomaly* balances the β-induced force. The first question that comes to mind is why only the mass anomaly is involved, and not the mass of the vortex as a whole. The answer is that water surrounding the vortex has to move in the direction opposite to that of the vortex, leading to a Coriolis force on that part of the layer that is pointing in the opposite direction. The net effect is a Coriolis force on the mass anomaly:

$$(f_0 + \beta Y) \frac{dX}{dt} \int_{\text{area}} \eta \, dA = (f_0 + \beta Y) \frac{dX}{dt} \int_{\text{area}} h \, dA - (f_0 + \beta Y) \frac{dX}{dt} \int_{\text{area}} H \, dA, \tag{23}$$

in which the integral is taken over a finite area, much larger than the vortex diameter. For the meridional direction a similar expression can be formulated. Note that the mass anomaly is negative for cyclones; this will be discussed later.

We first notice that the propagation velocity in the meridional direction is zero, because, as Killworth (1983) mentions, a vortex translating steadily in the meridional direction implies relative vorticity or thickness changes from potential vorticity conservation, violating the assumption of steadiness. This also follows directly from our (22) since the northward mass transport in the vortex excluding the motion of the vortex as a whole has to be equal to the southward mass transport in the vortex excluding the meridional motion of the vortex as a whole. This follows directly from continuity. Hence the right-hand side of (22) is zero. This allows us furthermore to choose $Y = 0$, so that

$$V \frac{dX}{dt} f_0 = -\int \left(hu - \eta \frac{dX}{dt} \right) \beta y \, dA \quad \text{and} \tag{24}$$

$$\frac{dY}{dt} f_0 = 0. \tag{25}$$

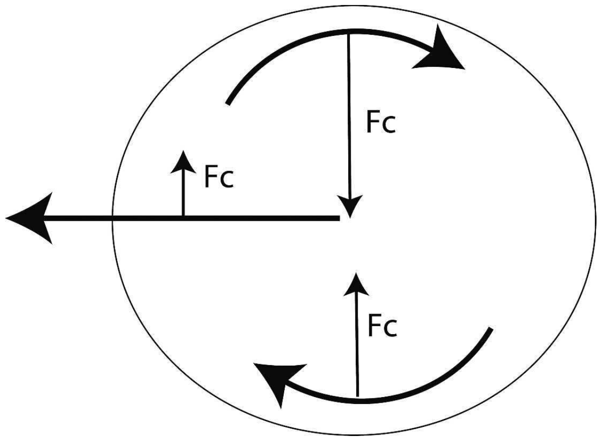

FIG. 1. Force balance on an anticyclone in the Northern Hemisphere. The extra Coriolis force on the plane due to the westward motion of the vortex is needed to resolve the force imbalance due to β.

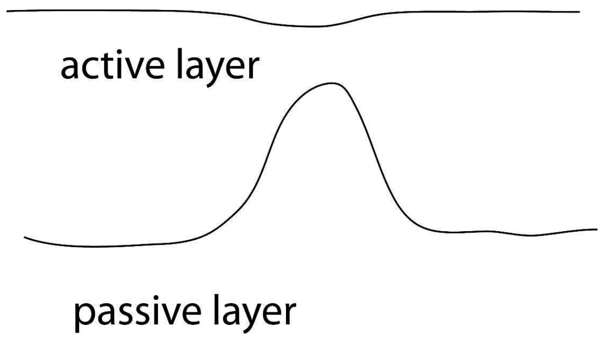

FIG. 2. The cyclone can be viewed as a hole in the active layer. Its mass anomaly is negative.

The right-hand side of (24) is negative for an anticyclone and positive for a cyclone (on the Northern Hemisphere). Because the mass anomaly of an anticyclone is positive and that of a cyclone is negative, both kind of vortices have to move westward. (An exception is an anomalous low that will be discussed in the concluding section of this paper.) This is independent of the exact shape of the vortex. For instance, even a small net mass anomaly in a dipolar field has to move westward. This is not a new result, but the following explanation is new.

The physical explanation of steady vortex motion is that the β-induced force on the layer is balanced by the Coriolis force on the mass *anomaly*. This is the reason why the vortex moves. In Fig. 1 the force balance for an anticyclone on the Northern Hemisphere is depicted. Because the Coriolis force on the mass transported eastward in the northern part of the vortex is larger than that on the equal amount of mass transported westward in the southern part of the vortex (the β effect), a net force on the vortex comes into play. For the anticyclone it is directed southward. If the acceleration of the vortex in the meridional direction is zero, as is the case here, a northward-directed force has to exist. This force is due to translation of the vortex as a whole. The water that moves with the vortex westward experiences a northward Coriolis force. The water that replaces the westward-moving water moves eastward, experiencing a southward Coriolis force. Because the mass inside the vortex is larger than the mass of the replacing water a northward Coriolis force remains. This force balances the force due to the variation of the Coriolis parameter.

For a cyclone on the Northern Hemisphere the swirling motion in the vortex is in the opposite direction, leading to a β-induced force pointing northward. So, for a cyclonic vortex that experiences no meridional acceleration, a southward-directed counterforce has to exist. A westward motion of the vortex as a whole leads to a northward Coriolis force on the vortex water. The water mass that has to replace the vortex water moves eastward and experiences a southward Coriolis force. Because the mass inside the vortex is smaller that of the replacing water, the net Coriolis force due to vortex motion is southward. This is exactly what is needed to balance the force due to β.

While the motion of the anticyclone can be considered as that of a positive mass anomaly, the motion of a cyclone can be viewed as the motion of a hole in the active upper layer (see Fig. 2). Indeed, the mass transport in a cyclone is eastward, while the vortex moves westward. Or, in other words, since the mass anomaly of a cyclone is negative, force and acceleration have opposite signs.

One of the reviewers mentioned that cyclones cannot move steadily westward because of direct coupling with Rossby waves. The appendix shows that the decay of simulated cyclones is typically a few percent of the vortex interface elevation amplitude during one rotation period within the cyclone, showing that the momentum balance discussed above can hold to a very good approximation.

On the Southern Hemisphere the rotation directions of the vortices change sign, and so do the β-induced forces. However, because the Coriolis force on a moving water column changes sign too, the balances only change sign and the westward motion remains.

The above explanation is the one of the main results of this paper. It is stressed that the formation of dipolar structures plays no role in this argument. Before an explanation of this latter fact is given, we discuss the literature in view of the above explanation.

The expressions by Nof (1981), Killworth (1983), Flierl (1984), and Dewar (1988) are true for lenses. The propagation mechanism is that the net Coriolis force on the swirling mass is compensated by the Coriolis force due to the bulk vortex translation. Indeed, the mass anomaly in their cases is the whole active layer, so the whole vortex. In our case the β-related Coriolis force is balanced by the Coriolis force due to the mass anomaly of the vortex only. The expression that the above authors find for the translation speed for cyclogeostrophic swirling motion is

$$\frac{dX}{dt} = -\frac{\beta}{Vf_0^2}\int\left(\frac{1}{2}g'\eta^2 + \frac{1}{2}hU^2\right)dA, \qquad (26)$$

which is the sum of available potential energy and kinetic energy of the lens divided by the mass of the lens. This expression differs from the steady form of our expression (20) by a term $-\beta R_d^2$. This term is of dominant order and is related to the β-related Coriolis force on the swirling motion of the vortex with layer thickness H. Because such a water mass will be present in realistic ocean vortices, the propagation speed of lenses (typically $2/3\beta R_d^2$; see, e.g., Nof 1981) is expected to be a factor of about 2 smaller than that encountered in real vortices.

To discuss the paper by Cushman-Roisin et al. (1990) we note that the first-order balance of the moving water columns is geostrophy, leading to [see Cushman-Roisin et al. (1990) and our equations in (14) and (15) in the steady limit]:

$$\frac{dX}{dt} = -\beta R_d^2\left(1 + \frac{1}{V}\int\frac{1}{2}\frac{\eta^2}{H}\,dA\right) \quad \text{and} \qquad (27)$$

$$\frac{dY}{dt} = 0. \qquad (28)$$

These equations show that both anticyclones and cyclones move westward, but the interpretation and explanation of these equations given by Cushman-Roisin et al. (1990) is unsatisfactorily. They argue that the westward motion of the vortex is due to two effects. The first is a net Coriolis force on the vortex. They argue that this force leads to westward motion for anticyclones and eastward motion for cyclones. Also, they argue that this effect is responsible for the second term in (27). However, as seen above, this is not true. The effect of the Coriolis force is given by the first term in (27), and it leads to westward motion for both anticyclones and cyclones.

They proceed by arguing that because of this initial motion a dipole appears in the surrounding waters that pushes the vortex westward, overruling the initial motion. The dipole comes into existence because of north- and southward displacement of water surrounding the vortex. According to them, this is the first term in (27). So, in fact they argue that a secondary effect, the formation of a dipole by the displacement of surrounding water, leads to a first-order adaptation of the initial vortex motion. This is not too satisfactory either because it is unclear where the boundary between vortex and surrounding water lies. Furthermore, the explanation needs a specific structure of vortex and surrounding water that is not in the equations above, which were integrated over the whole β plane. The explanation given the present paper does not have these drawbacks. In fact, the second term in (27) is due to the fact that the mass anomaly also plays a role in the β-induced force, increasing it in the case of anticyclones and decreasing it for cyclones.

Nof (1983) derives an expression for the motion of a vortex by considering an inner domain that translates with the vortex with velocity C, and an exterior domain.

He considers the equations in a coordinate system moving with the vortex and defines a streamfunction in this system. The vortex edge is taken as a streamline of the exterior field. By integrating the meridional momentum equation over the vortex area he finds, in our notation,

$$\int_{\text{vortex}}(f_0 + \beta y)Ch\,dA = -\beta\int_{\text{vortex}}\psi\,dA + \frac{g'}{2}\oint h^2\,dS, \qquad (29)$$

in which the integrals are over the inner domain and its boundary, respectively. Here ψ is a streamfunction, defined by $hu = -\psi_y$ and $hv = \psi_x$, which is chosen to be zero along the boundary. The advection terms vanish because of the choice of the vortex edge as being a streamline. In this equation, apart from the β-induced force, a pressure force over the boundary of the vortex arises. This pressure force is due to the exterior flow, and depends on the size of the inner domain and the exterior velocities at the boundary. Vortex motion is now due to a balance between the Coriolis force on the water columns moving with the vortex, the β-induced force due to rotation inside the vortex and this pressure force at the boundary. This latter force is coined planetary drift by Nof (1983). Equation (29) is interesting because it uses the natural division between inner and outer vortex areas. The dividing streamline is the socalled separatrix (see, e.g., De Steur et al. 2004), which is the closed streamline that connects to infinity. The separatrix has a singular point where it crosses itself, and is not circular, but that does not invalidate the above expression for the propagation speed to first order.

The expression by Nof (1983) has a few drawbacks. First, the speed of the vortex seems to be related to the size of the vortex, while our expression shows that it does not. Second, to find meaningful expressions for the speed of cyclones and anticyclones more complicated perturbation expansions are needed (see Nof 1983).

To understand the relation with our expression in (24) we first evaluate the pressure term. Because the pressure is continuous over the edge of the vortex, the last integral is equal to the integral of the layer thickness squared of the exterior field. The momentum equation integrated over the exterior field can then be used to help our interpretation:

$$\frac{g'}{2}\oint h^2\,dS = \frac{g'}{2}\oint h_{\text{ex}}^2\,dS = -\int_{\text{ex}} (f_0 + \beta_y)(C + u)h\,dA. \tag{30}$$

Again the advection terms cancel because of the choice of the vortex edge. We thus find that the "planetary drift" term is just the Coriolis force on the exterior field. In order for the terms on the right-hand side to be finite, u should become equal to $-C$ far from the vortex center, where $h \approx H$. This just means that the water does not move in the coordinate system attached to the rotating earth. If we combine the last two equations we arrive at

$$\int_{\text{vortex}} (f_0 + \beta y)Ch\,dA = -\beta\int_{\text{vortex}} \psi\,dA - \int_{\text{ex}} (f_0 + \beta y)(C + u)h\,dA. \tag{31}$$

Now we see that vortex motion is due to a balance between the Coriolis force on the water columns moving with the vortex, the β-induced force due to the circulation in the vortex, and the Coriolis force on the exterior flow. This latter flow is needed to compensate for the westward motion of the body of the vortex.

The connection with the explanation given above is found when the boundary is extended to infinity. First the β-related term that contains the streamfunction is rewritten in terms of hu because the streamfunction definition is connected to the vortex edge. We then find that

$$\int_{\text{vortex}} (f_0 + \beta y)Ch\,dA = -\int_{\text{vortex}} (f_0 + \beta y)uh\,dA - \int_{\text{ex}} (f_0 + \beta y)(C + u)h\,dA, \tag{32}$$

in which "vortex" now means integration over the whole β plane. The exterior contribution vanishes because $C + u$ becomes zero far from the vortex center. Furthermore, if the coordinate system to the system fixed to the earth is changed, $u_{\text{fixed}} = C + u$, then (24) is recovered. Now that we have found the connection with previous expressions and explanations it is time to turn to the question of what the influence of dipolar motions is on the propagation of vortices.

b. *Dipolar vortices in steady vortex motion*

One may wonder what the contribution of a dipolar field is to the motion of the compensated vortex. No doubt, dipolar fields are present on a β plane (see, e.g., Nof 1981), but the question is what their role is in the propagation of a vortex. At first glance, one would expect a contribution due to the advection of the center of the vortex in the direction of the dipole's main axis, but such a term is not present in the explanation given above. Furthermore, the paper by Nycander and Sutyrin (1992) contains examples in which the dipole is oriented such that it decreases the westward motion of the vortex, while the general idea is that dipoles enhance the westward migration of monopoles. In the following two points of view are explored to obtain more insight in these matters.

We start from (6) and splits the velocity field in a part u_d that is symmetric with regard to $y = y_0$ and a part u_m that is antisymmetric with regard to $y = y_0$. (Note that $y_0 \neq 0$ because it is not equal to the center of mass in the meridional direction, which is shifted meridionally because of the dipolar field.) Clearly, the monopolar field is contained in u_m, while a dipolar contribution resides in u_d. Equation (6) now becomes

$$V\frac{dX}{dt} = \int u_d h\,dA + \int u_m h\,dA. \tag{33}$$

From symmetry arguments we obtain

$$V\frac{dX}{dt} = \int u_d h_{\text{nond}}\,dA + \int u_m h_{\text{nonm}}\,dA, \tag{34}$$

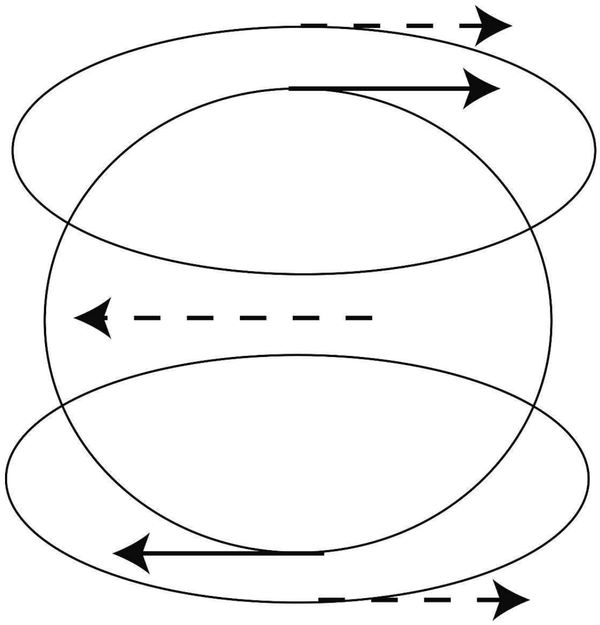

FIG. 3. Schematic view of the interaction between the anticyclone and the dipole. The dipole tends to advect the monopole westward (dashed arrows). The monopole advects the anticyclonic part of the dipole eastward and its cyclonic part westward (solid arrows), leading to a cumulative eastward mass transport. The total mass transport is small and westward.

in which $h_{\rm nond}$ is the mass field that is not exactly dipolar, and $h_{\rm nonm}$ is the mass field that is not exactly monopolar. Because of the nonlinearity of the flow field, "nond" does not automatically mean m. The first term describes the advection of the nondipolar mass field by the dipolar velocity field. This term is referred to when the influence of dipoles on vortex motion is discussed. The second term describes the advection of the nonmonopolar mass field by the monopolar velocity field. This second term, often overlooked, is also important. An anticyclone with a zonally oriented dipole on the Northern Hemisphere is depicted (see Fig. 3). The dipole consists of an anticyclone north of the center of the vortex and a cyclone to the south. Clearly, the mass anomaly of the monopole is advected westward by the dipolar velocity field. This leads to a westward motion of the mass anomaly. On the other hand, the mass anomaly of the northern anticyclone of the dipole is advected eastward by the monopolar velocity field, while the mass anomaly of the southern cyclone of the dipole is advected westward. The net result of this is an eastward motion of the center of mass of the dipole (see Fig. 3). So, the mass advection by the monopolar velocity field counteracts that by the dipolar velocity field. If the dipole is rotated 180° the dipole advects the mass anomaly of the monopole eastward, while the monopole advects the dipolar mass anomaly westward. Nycander and Sutyrin (1992) show an example of this configuration. It is difficult to determine from the above analysis which effect is dominant. It is easy to see that both terms must be of the same order of magnitude, because the first-order velocity fields will be geostrophic, leading to an exact cancellation. So, in short, the action of the monopole on the dipole should not be neglected.

Second, we turn to the expression given by Nof (1983):

$$\int_{\rm vortex}(f_0+\beta y)Ch\,dA=-\beta\int_{\rm vortex}\psi\,dA+\frac{g'}{2}\oint h^2\,dS. \tag{35}$$

The action of the dipole is present in the pressure term integrated over the vortex edge. We first notice that this action is not of an advective nature because the edge is a streamline. Furthermore, we have seen that this term is in fact equal to the Coriolis force on the mass flow outside the vortex. So, the dipole field is fully determined by the return flow outside the vortex, which is only a function of the size and shape of the monopole. It is not the dipole field that determines the velocity of a vortex, the dipole field strength is fully determined by the monopole. Expressions like "the vortex is advected by the dipole field" interchange course and effect.

Last, it is stressed that the above only holds for steadily moving vortices. When an arbitrary dipole field is added to the velocity field of a steadily moving vortex the momentum advection through the vortex boundary and the pressure over that boundary cannot simply be written as the Coriolis force on the exterior flow, and an acceleration of the vortex will occur.

3. Advection by a background flow

In this section we study the influence of a steady background current on vortex propagation, in a reduced-gravity model. We assume that the flow has horizontal dimensions far greater than that of the vortex, such that we can neglect its spatial variations in the neighborhood of the vortex. With this assumption this current is in geostrophic balance:

$$-f\overline{v}=-g'\overline{h}_x \quad \text{and}$$

$$f\overline{u}=-g'\overline{h}_y, \tag{36}$$

in which $f = f_0 + \beta y$. Because of the β plane the large-scale pressure gradient is not constant in space, while the current is constant. Because of the nonlinear equations of motion, it is impossible to separate the background current from the vortex motion when the latter is vigorous. A simple way to circumvent this difficulty is to define the background flow as being in geostrophic balance with the large-scale pressure gradient everywhere. As we will see in the following, this will not influence our final result, but it clarifies the derivation. We define the mass (or rather volume) anomaly of the vortex as

$$V = \int \eta \, dA, \tag{37}$$

in which $\eta = h - \overline{h}$ is the mass anomaly related to the vortex. (However, see the discussion above.) Since no exchange of mass is allowed with the lower layer, the mass anomaly is conserved as before. For convenience we define a perturbation velocity related to the swirling vortex motion and its propagation as $u' = u - \overline{u}$ and $v' = v - \overline{v}$. As before, the vortex position is defined as

$$X = \frac{1}{V}\int x\eta \, dA \quad \text{and}$$

$$Y = \frac{1}{V}\int y\eta \, dA, \tag{38}$$

from which we derive the propagation speed of the vortex as

$$V\frac{dX}{dt} = \int x \frac{\partial \eta}{\partial t} dA = V\overline{u} + \int hu' \, dA \quad \text{and} \tag{39}$$

$$V\frac{dY}{dt} = \int y \frac{\partial \eta}{\partial t} dA = V\overline{v} + \int hv' \, dA, \tag{40}$$

where we used the full continuity equation first, then that of the mean flow $\overline{u}\overline{h}_x + \overline{v}\overline{h}_y = 0$, followed by partial integrations. From this expression one might get the impression that the background current can just be added to the propagation speed of the vortex without background flow. However, that is incorrect because the background flow is still present in the integrals via the layer thickness. To evaluate this further, we take the time derivative of (39) and (40), and use the momentum equations to find for the balance in the zonal direction:

$$V\frac{d^2X}{dt^2} = \int \frac{\partial}{\partial t}(hu') \, dA = -\int (huu')_x + (hvu')_y \, dA + f_0 \int hv' \, dA + \beta \int yhv' \, dA - g' \int h\eta_x \, dA. \tag{41}$$

Because the background current is geostrophic the Coriolis and pressure gradient terms related to this flow are cancelled.

The first integral at the right-hand side of this expression is zero because u' and v' are zero far from the vortex center. The second integral is related to the meridional propagation speed of the vortex as found above. A partial integration of the last term leads to

$$-g' \int h\eta_x \, dA = g' \int \overline{h}_x \eta \, dA, \tag{42}$$

so that

$$V\frac{d^2X}{dt^2} = f_0 V \frac{dY}{dt} - f_0 V\overline{v} + g' \int \overline{h}_x \, \eta \, dA + \beta \int yhv' \, dA. \tag{43}$$

For the meridional direction we similarly find

$$V\frac{d^2Y}{dt^2} = -f_0 V \frac{dX}{dt} + f_0 V\overline{u} + g' \int \overline{h}_y \eta \, dA - \beta \int yhu' \, dA. \tag{44}$$

To complete the solution we have to evaluate the β-related term. Because of the nonlinearity of the full velocity near the vortex, it is impossible to separate the contribution of the background current. However, a very good order-of-magnitude estimate can be obtained when we assume that the velocities are geostrophic in these β terms. With that assumption, we can make a distinction between background and vortex motion as follows:

$$\begin{aligned}\beta \int yhv' \, dA &= \frac{g'\beta}{f_0}\int y\overline{h}\eta_x \, dA + \frac{g'\beta}{f_0}\int y\eta\eta_x \, dA \\ &= -\frac{\beta}{f_0} g' \int y\eta\overline{h}_x \, dA \\ &= -\beta Y\overline{v}V,\end{aligned} \tag{45}$$

where we neglected terms proportional to β^2. For the other direction we find

$$\begin{aligned}-\beta \int yhu' \, dA &= \frac{g'\beta}{f_0}\int y\overline{h}\eta_y \, dA + \frac{g'\beta}{f_0}\int y\eta\eta_y \, dA \\ &= -\beta Y\overline{u}V - V\frac{\beta g' H}{f_0} - \frac{\beta g'}{2f_0}\int \eta^2 \, dA,\end{aligned} \tag{46}$$

in which we again neglected terms proportional to β^2 and in which we defined the still-water depth at the position of the vortex as

$$H = \frac{1}{V}\int \overline{h}\eta \, dA. \tag{47}$$

Combining this with the momentum equations we finally obtain

$$V\frac{d^2X}{dt^2} = f_0 V\frac{dY}{dt} - (f_0 + \beta Y)V\overline{v} + g'\int \overline{h}_x \eta \, dA, \tag{48}$$

which simplifies to

$$V\frac{d^2X}{dt^2} = f_0 V\frac{dY}{dt}. \tag{49}$$

For the meridional direction we similarly find

$$V\frac{d^2Y}{dt^2} = -f_0 V\frac{dX}{dt} + (f_0 + \beta Y)V\overline{u} + g'\int \overline{h}_y \eta \, dA - V\frac{\beta g' H}{f_0} - \frac{\beta g'}{2f_0}\int \eta^2 \, dA, \tag{50}$$

which reduces to

$$V\frac{d^2Y}{dt^2} = -f_0 V\frac{dX}{dt} - V\beta\frac{g'H}{f_0} - \frac{\beta g'}{2f_0}\int\int \eta^2 \, dA. \tag{51}$$

These expressions for the propagation speed of the vortex are identical to expressions obtained without background flow. Hence, a background flow does not advect an ocean vortex in a reduced-gravity model, at least not to first order. This is another main result of this paper.

The reason for this counterintuitive result is readily obtained form the derivation given above. With the integration over the β plane all terms related to the advection of the vortex by the background current vanish because the vortex-related motion is negligible far from the vortex core. In this description of the vortex motion momentum, advection cannot play a role. The background flow can only influence the vortex motion by a Coriolis force on the mass anomaly by advection with the background flow, and by a pressure gradient on the mass anomaly. However, because the background flow is in geostrophic balance far from the vortex core, these two terms cancel to first order, leaving us with a vortex motion similar to that without the background flow.

Another explanation can be obtained from (50). We take this equation in its steady-motion form for clarity. We then have, after division by $f_0 V$ and neglecting the η^2 and the $\beta\overline{h}_y$ term,

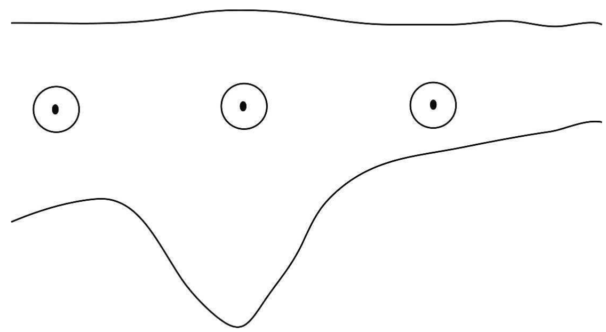

FIG. 4. Vertical north–south section of the trough of the active layer when a background current (open circles) is present. The current is eastward, but the sloping interface induces a topographic β effect and hence a westward-propagation speed of the vortex. The advection and the topographic β effect cancel each other to first order.

$$\frac{dX}{dt} = -\beta\frac{g'H}{f_0^2} + \overline{u} + \frac{g'}{f_0}\overline{h}_y, \tag{52}$$

which can be rewritten as

$$\frac{dX}{dt} = -\beta R_d^2 + \frac{f_0}{H}\overline{h}_y R_d^2 + \overline{u}. \tag{53}$$

The reasoning is now as follows. The geostrophic background flow is accompanied by a background layer thickness gradient. This gradient acts as a background potential vorticity gradient for the vortex, and has a similar effect as β in modifying the propagation speed of the vortex. (It reminds us of topographic Rossby waves.) Hence the advection of the mass anomaly by the geostrophic background current is cancelled by the vortex speed change due to the change in ambient potential vorticity. This mechanism is depicted in Fig. 4. It is similar to that of a background flow on long Rossby waves in a reduced-gravity model, whose speed is also not changed (see, e.g., Killworth et al. 1997).

4. Vortex propagation starting from rest

In this section, we study the initial propagation of a vortex when it starts from rest. The governing equations for the motion of the center of mass of the vortex in a reduced-gravity model read as follows:

$$V\frac{d^2X}{dt^2} = V\frac{dY}{dt}(f_0 + \beta Y) + \int\left(hv - \eta\frac{dY}{dt}\right)\beta y \, dA \quad \text{and} \tag{54}$$

$$V\frac{d^2Y}{dt^2} = -V\frac{dX}{dt}(f_0 + \beta Y) - \int\left(hu - \eta\frac{dX}{dt}\right)\beta y \, dA. \tag{55}$$

Assuming geostrophic flow for the swirl velocities, $\beta Y \ll f_0$, and $\eta < H$, as first approximation, we obtain

$$\frac{d^2X}{dt^2} = f_0\frac{dY}{dt} \quad \text{and} \tag{56}$$

$$\frac{d^2Y}{dt^2} = -f_0\frac{dX}{dt} - f_0\beta R_d^2. \tag{57}$$

With an initial propagation speed of zero, we find that the vortex will accelerate *equatorward* first, to turn westward later. This result is independent of the sign of the vortex (i.e., both anticyclones and cyclones move this way initially). The reason for this behavior is that the Coriolis force on the poleward side of the vortex is larger than that on the equatorward part. (Note that the total westward mass transport is equal to the total eastward mass transport by definition of zero center-of-mass velocity.) This imbalance of the Coriolis force on the vortex leads to an acceleration in the equatorward direction. For instance, an anticyclone on the Northern Hemisphere has eastward flow in the northern half and westward flow in its southern half. The Coriolis force on the northern part is directed southward, while that on the southern half is directed northward. Since the former is larger, a southward net force, and hence an acceleration results.

We also discuss a cyclone in the Northern Hemisphere because the general idea is that it tends to move poleward. This has to do with the fact that a northward motion tends to reduce the potential vorticity anomaly with respect to the surroundings. A similar reasoning as for the anticyclone gives rise to a net northward-directed net Coriolis force on the vortex. However, since the mass anomaly in the cyclone is negative, the acceleration is also southward.

The formal solution of (56) and (57) is found as

$$\frac{dX}{dt} = -\beta R_d^2(1 - \cos f_0 t) \quad \text{and}$$

$$\frac{dY}{dt} = -\beta R_d^2 \sin f_0 t, \tag{58}$$

that is, a sinusoidal inertial oscillation on top of a westward translation. When we calculate the trajectory of the ring, assuming $X(0) = 0$ and $Y(0) = 0$ we find

$$X = -\beta R_d^2 t + \frac{\beta R_d^2}{f_0}\sin f_0 t \quad \text{and}$$

$$Y = \frac{\beta R_d^2}{f_0}(\cos f_0 t - 1), \tag{59}$$

from which we see that the amplitude of the oscillational motion is $\beta R_d^2/f_0 \approx 200$ m. This is negligible for ocean rings: hard to detect in situ and difficult to model.

These results seems to be in contradiction with the findings of Adem (1956), Reznik and Dewar (1994), Benilov (2000), and Reznik and Grimshaw (2001), among others. We will discuss this discrepancy in light of the work of Adem (1956), and then generalize the results to the other papers. Using a initially circular symmetric cyclone in a barotropic quasigeostrophic model, Adem found that the barotropic cyclone first moves westward, and develops a poleward component afterward. The difference with our description lies in the initial condition, and the barotropic assumption. Adem (1956) uses a Taylor series expansion in time to obtain the following expression for the streamfunction evolution:

$$\Psi(r, \theta, t) = \psi(r) + \beta\cos\theta\left(F_1 t + \frac{1}{3!}F_3 t^3 + \cdots\right) - \beta\sin\theta\left(\frac{1}{2!}F_2 t^2 + \frac{1}{4!}F_4 t^4 + \cdots\right), \tag{60}$$

in which F_i are functions of r, dependent on the shape of the initial circular streamfunction $\psi(r)$. The center of mass is found as

$$X = \frac{1}{V}\int x\eta\, dA = \frac{\int x\Psi\, dA}{\int \psi\, dA} = \beta\frac{\int \cos^2\theta\, d\theta \int\left(F_1 t + \frac{1}{3!}F_3 t^3 + \cdots\right)r^2\, dr}{\int \psi\, dA} = \frac{1}{2}\beta\frac{\int\left(F_1 t + \frac{1}{3!}F_3 t^3 + \cdots\right)r^2\, dr}{\int\psi\, dA} \tag{61}$$

and, similar,

$$Y = \frac{1}{V}\int y\eta\, dA = \cdots = \frac{1}{2}\beta\frac{\int\left(\frac{1}{2!}F_2 t^2 + \frac{1}{4!}F_4 t^4 + \cdots\right)r^2\, dr}{\int\psi\, dA}. \tag{62}$$

When we calculate the velocity of the center of mass we find

$$\frac{dX}{dt} = \frac{1}{2}\beta\frac{\int\left(F_1 + \frac{1}{2!}F_3t^2 + \cdots\right)r^2\,dr}{\int\psi\,dA} \quad \text{and}$$

$$\frac{dY}{dt} = \frac{1}{2}\beta\frac{\int\left(F_2t + \frac{1}{3!}F_4t^3 + \cdots\right)r^2\,dr}{\int\psi\,dA}. \tag{63}$$

We thus see that the center of mass of the initial vortex *is moving westward at initial time* in the solution of Adem (1956) because (note the misprint in Adem's paper):

$$F_1 = -\frac{1}{r}\int_0^r \psi(r)r\,dr > 0, \tag{64}$$

and the denominator is smaller than zero, so his initial condition is different from ours. The above result is independent of the exact shape of the vortex, as long as it is circular symmetric. The deeper reason is that Adem (1956) assumes a circular pressure field at initial time. On a β plane this gives rise to a noncircular velocity field, with larger velocities in the northern half of the cyclone. Since the velocities in the northern half of the cyclone are directed westward, a westward motion of the cyclone results. So, in this case the total westward mass transport is not equal to the total eastward mass transport at initial time. Also Reznik and Dewar (1994), Benilov (2000), and Reznik and Grimshaw (2001) start with a radially symmetric initial pressure field, and the explanation given above holds also for these and similar papers.

The exact correspondence with the force balance on a barotropic vortex is difficult to obtain. The reason is the extremely large external Rossby radius for oceanographic applications, about 2×10^6 m, leading to an enormous southward acceleration in (56). Furthermore, a steady-state solution would have an enormous westward-propagation speed of $-\beta R_d^2$, which is about 80 m s^{-1}. Clearly, this is highly unrealistic. In fact, a steady-state solution in a barotropic quasigeostrophic context is only consistent with a zero mass anomaly of the vortex, as found by Stern (1975). Numerical model results indicate that a barotropic vortex in a barotropic flow experiences strong filamentation and Rossby wave emission as found by L. De Steur (2005, personal communication). It seems that the mass anomaly is smeared out over a large area, but the exact evolution of the system is unclear because of interaction with the boundaries of the domain in the numerical simulations.

5. Conclusions and discussion

This paper discusses the motion of monopolar vortices in a reduced-gravity model. Expressions for the center-of-mass propagation for vortices on a β plane have been derived. A consistent physical explanation is given for the steady westward motion of vortices, both anticyclonic and cyclonic. This motion is needed to generate a Coriolis force on the center of mass of the vortex that compensates for the imbalance of the integrated Coriolis force over the swirling motion of the vortex, which is due to the β effect. Essential for cyclones is that their mass anomaly is negative, so that force and acceleration have opposite signs. The relation to other explanations presented in literature is discussed. It is argued that lens models tend to underestimate the propagation speed of vortices by about a factor of 2 because these ignore the swirling motion of the water mass that is not connected with the mass anomaly of the vortex. Dipolar fields can accelerate vortices in general, but it is shown that they must have a very specific structure for a steadily propagating vortex, and their net effect on the propagation speed is small. It has been put forward that steady westward-moving cyclones do not exist because of Rossby wave coupling (see, e.g., Nycander 1994; Benilov 1996). Numerical experiments described briefly in the appendix support this claim. However, the interface decay during one rotation around the vortex core is typically a few meters (i.e., a few percent) showing that the first-order momentum balance is as described here.

The potential advection of vortices by a geostrophic background flow is also discussed. It is shown that the propagation speed of a vortex does not change with the inclusion of such a flow in a reduced-gravity model. The reason for this counterintuitive effect is that the advection by the flow is counteracted by the change in the background potential vorticity field in which the vortex moves because of the sloping background interface.

Last, we showed that vortices in a reduced-gravity model move equatorward first when starting from rest, after which a westward motion develops. This result also holds for cyclones. The discrepancy with the results of Adem (1956) and others are discussed and shown to be related to the different initial conditions: in Adem (1956), as well as in the other papers the center of mass of the cyclone already moves westward at the initial time.

All results have been based on the assumption that the center-of-mass velocity is a good representation of the actual motion of the vortex (e.g., the motion of the interface maximum). It is shown in the appendix that the center of potential vorticity moves with the same speed, a result that has been derived by Larichev (1984) in the quasigeostrophic model. Furthermore, numerical experiments show that the deviation between the center of mass and the interface maximum is typically smaller than the vortex diameter after 500 days of integration, both for anticyclones and for cyclones. These results let us to the conclusion that the center-of-mass description is a useful one for vortex propagation.

We are interested in what happens when the restriction to reduced gravity is relaxed. If a vortex is not compensated a new driving mechanism appears that can have a substantial effect on the motion of the vortex. The mechanism is due to the fact that the pressure terms do not integrate out, so that a pressure gradient force is entering the force balance. If the sea surface elevation is given by ξ, these terms are $g\langle\eta\xi_y\rangle$ and $g\langle\eta\xi_x\rangle$. Because the terms are not proportional to β they can be very effective in changing the magnitude and even the direction of the vortex relative to the compensated case (see, e.g., Cushman-Roisin et al. 1990; Chassignet and Cushman-Roisin 1991). This argument clearly illustrates that the deeper layers can be of vital importance in our understanding of vortex motion.

The above pressure terms might give rise to an erroneous interpretation of their physical effect in the following way. They can be rewritten as $\langle\eta(g\xi_y - g'\eta_y)\rangle$, which is equal to $\langle\eta p_{2_y}/\rho_0\rangle$, in which p_2 is the pressure at the interface that drives motions in the second layer. If ξ and η are displaced in the meridional direction, p_2 will have a dipole structure. This dipole structure will lead to a dipolar velocity field in the second layer through geostrophy. It is sometimes argued (see, e.g., Herbette et al. 2003) that this velocity field will push the interface bowl into the direction of that field. The flaw in this line of reasoning is that since friction at the interface is neglected, the velocity field in the second layer is of no direct influence to that in the upper layer. It is similar to the d'Alambert paradox, which shows that an object in a flowing fluid experiences no net force when friction is neglected.

One can argue that the β-plane equations are not suitable to study vortex motion because it is unclear if westward motion means along parallels or along a great circle (see Van der Toorn 1997). Graef (1998) studied the influence of several extensions of the β plane to include curvature terms in a lens model. He found no change in the equations for the steady propagation speed of the center of mass of the lens. Nycander (1994) and McDonald (1998) discuss the motion of a solid spinning disk on a rotating planet, and find similar equations for the propagation speed. In our next paper the motion of a vortex on the sphere will be investigated in a coordinate system attached to the rotating earth in which the active layer does not vanish at infinity.

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APPENDIX

The Center-of-Mass Velocity as Vortex Propagation Speed

One can argue that the center of mass is not a good measure of the propagation speed of the vortex because of, for example, Rossby wave radiation, vortex filaments, and the creation of smaller satellite vortices. Indeed, as direct oceanographic observations and numerical experiments show, these processes appear, and might disturb the picture of a propagating monopole. The problem is that all integrated measures like the center of mass cannot distinguish between vortex motion and all other features. On the other hand, evolution equations for local measures like the maximum interface elevation or the maximum of the relative velocity show strong dependence on the detailed vortex structure, which is difficult to obtain from reality. The strength of the integrated measures is that they integrate out all these details and try to concentrate on the picture at large.

In this appendix two arguments are put forward as to why the center-of-mass velocity is a reasonable measure for the propagation of the main vortex. In the first an expression for an approximation of the area-integrated potential vorticity evolution is derived. It is shown that this expression is equal to the one from the center of mass. In the second argument numerical experiments are presented that show that the center-of-mass velocity and the maximum of the interface elevation velocity are the same to first order.

Let us concentrate on the evolution of the area-integrated anomalous potential vorticity, given by

$$\pi = \frac{\zeta + f}{h} - \frac{f}{H}, \tag{A1}$$

which is just the potential vorticity minus the background planetary contribution. This other measure of the vortex position will give more insight into the robustness of the center-of-mass description.

We define the zonal position of the vortex as

$$X_v = \frac{\int x\pi h\, dA}{\int \pi h\, dA} \tag{A2}$$

and a similar equation for the meridional position. The vertical integration, which leads to the factor h, ensures that we use the potential vorticity anomaly of the whole column. To find the rate of change of this position we first concentrate on the denominator. We have

$$\frac{d}{dt}\int \pi h\, dA = \int \frac{d\pi}{dt} h\, dA = -\int \beta v \frac{h}{H}\, dA, \tag{A3}$$

where we know that $d(hdA)/dt = 0$ in a Lagrangian interpretation of the integral. This expression shows that the denominator in (A2) does not depend on time to the first order, which is geostrophic. The denominator itself can be evaluated as

$$\int \pi h\, dA = \int \left(\zeta + f - \frac{h}{H} f\right) dA = -\frac{1}{H}\int \eta f\, dA. \tag{A4}$$

For the numerator we find

$$\begin{aligned}\frac{d}{dt}\int x\pi h\, dA &= \int u\pi h + x\frac{d\pi}{dt} h\, dA \\ &= -\int u v_x\, dA - \frac{1}{H}\int u\eta f\, dA \\ &\quad - \int x\beta v \frac{h}{H}\, dA.\end{aligned} \tag{A5}$$

An order of magnitude of these expressions can be obtained by assuming geostrophy. The first term on the right-hand side of the equation above integrates to zero after two integrations by parts. For the other terms we find

$$\begin{aligned}\frac{d}{dt}\int x\pi h\, dA &\approx -\frac{g'\beta}{2f_0 H}\int \eta^2\, dA + \frac{\beta g'}{f_0}\int \eta\, dA \\ &\quad + \frac{\beta g'}{2f_0 H}\int \eta^2\, dA = \frac{\beta g'}{f_0}\int \eta\, dA.\end{aligned} \tag{A6}$$

Combining the numerator with the denominator in (A2) we find

$$\frac{dX_v}{dt} \approx -\beta R_d^2, \tag{A7}$$

in which we neglected the βy term with respect to f_0. This expression is identical to the first-order expression for a vortex obtained from the center-of-mass velocity. It is clear that the details in the two measures for vortex propagation speed will differ at a higher order, but the fact that the two expressions are identical to leading order gives credit to the center-of-mass expressions. Larichev (1984) shows the same result by using the barotropic quasigeostrophic potential vorticity, which is now generalized to the full shallow-water potential vorticity. This is backed up further by numerical experiments.

Several numerical experiments have been performed with a reduced-gravity ocean model with grid spacing of 10 km, a domain size of 200 × 200 grid points, and an undisturbed layer depth of 500 m. The central latitude ranged from 20° to 50°N, ring diameters ranged from 2 to 4 times the Rossby radius of deformation, and interface elevation maxima ranged from 0.1 to 0.9 times the undisturbed water depth. The experiments were carried out both for anticyclones as well as for cyclones, with Gaussian radial interface profiles, and profiles with a solid-body rotation at the core:

$$\begin{aligned} h &= A\left(r_0^2 - \frac{4}{3}r^2\right) && \text{for } r \le r_m \\ &= A\left(r_0^2 - \frac{4}{3}r^2\right)\exp\left[-\frac{2}{3}\left(\frac{r^3}{r_0^3} - \frac{r_m^3}{r_0^3}\right)\right] && \text{for } r > r_m \end{aligned}. \tag{A8}$$

In some experiments dipolar moments of the following form have been added:

$$\eta = A\left[1 - 0.3\frac{(y - y_0)}{r_0}\right]\exp\left(-\frac{r^2}{2r_0^2}\right). \tag{A9}$$

It turns out that the profiles tested have little influence on the propagation speed and direction of the vortex, and on its decay. All anticyclones had a zonal velocity a bit faster than βR_d^2, and all cyclones moved a bit slower. Typically, the difference in position of the elevation maximum (or minimum) and the center of mass was always smaller than one ring diameter after 500 days, showing that the center-of-mass approximation of the propagation speed for an oceanic vortex is very good.

All vortices showed an elevation maximum decay due to Rossby wave radiation. This lead to equatorward motion of anticyclones, and to poleward motion of cyclones. As mentioned above, these meridional motions are relatively mild in comparison with the size of the vortices. Anticyclones tended to a state of zero decay after 500 days, approaching pure zonal motion, while cyclones kept decaying slowly at a rate of about 10 m (100 days)$^{-1}$ and moving poleward at a rate of about 15 km (100 days)$^{-1}$. These findings are in line with Nycander (1994) and Benilov (1996), who argue that cyclones cannot move steadily because their zonal propagation speed is smaller than that of Rossby waves, leading to strong coupling and thus decay by wave radiation. However, the numerical results show that the cyclones do move with constant speed, and that the decay is mild. Assuming a swirl velocity of 20 cm s^{-1} and a radius of 90 km for a cyclone, a water parcel will need about 1 month to make one loop, in which the elevation minimum reduces by a few meters. This conservative estimate shows that the decay can be neglected to first order, and a nearly steady motion of a cyclone in a reduced-gravity model is a reasonable assumption.

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