On the Steadiness of Separating Meandering Currents

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ABSTRACT

The existence of inertial steady currents that separate from a coast and meander afterward is investigated. By integrating the zonal momentum equation over a suitable area, it is shown that retroflecting currents cannot be steady in a reduced gravity or in a barotropic model of the ocean. Even friction cannot negate this conclusion. Previous literature on this subject, notably the discrepancy between several articles by Nof and Pichevin on the unsteadiness of retroflecting currents and steady solutions presented in other papers, is critically discussed. For more general separating current systems, a local analysis of the zonal momentum balance shows that given a coastal current with a specific zonal momentum structure, an inertial, steady, separating current is unlikely, and the only analytical solution provided in the literature is shown to be inconsistent. In a basin-wide view of these separating current systems, a scaling analysis reveals that steady separation is impossible when the interior flow is nondissipative (e.g., linear Sverdrup-like). These findings point to the possibility that a large part of the variability in the world's oceans is due to the separation process rather than to instability of a free jet.

1. Introduction

In the World Ocean several places exist where large-scale ocean currents retroflect (i.e., make an anticyclonic turn of more than 90°) after separation. Examples are the Agulhas Current, the North Brazil Current, the Brazil Current, and the East Australian Current. A common observed feature of all these systems is that they are unsteady and shed rings. In a series of papers (e.g., Nof and Pichevin 1996, hereafter NP) state that under a rather restricting set of conditions, this is a necessity arising from a momentum imbalance for a steady frictionless retroflecting current (NP; Pichevin and Nof 1996, 1997; Nof and Pichevin 1999; Pichevin et al. 1999; see also Nof et al. 2004 for an interesting review). By integrating the zonal momentum equation over an area that contains a steady retroflecting current and making some assumptions on the in- and outflow, they derive a momentum imbalance. The imbalance itself is a remarkable result because the details of the retroflecting current do not matter.

Direct numerical simulations of retroflecting currents show that they are unsteady (e.g., T. Pichevin 2001, personal communication). However, such studies cannot be used to prove that steady retroflections cannot exist because forward integrations can never probe the state space in enough detail: the regions of attraction of possible steady states might be rather small. In fact, the steady state might be unstable, as Dijkstra and de Ruijter (2001) showed by using a continuation technique that follows steady states in parameter space. By forward numerical integration, such unstable steady states are impossible to find. On the other hand, Moore and Niiler (1974) and, in a more idealized setting, Ou and de Ruijter (1986) present steady solutions for retroflecting currents. So, although the idea of NP is appealing, it seems to be contradicted by other studies.

In the present paper (section 2), it is shown that a momentum imbalance theorem *can* be formulated for retroflecting flows. We show that the derivation as proposed by NP is valid only for currents that satisfy very specific outflow conditions, as detailed in the appendix. The difference with NP is that we treat all possible configurations that retroflecting currents can have and that we extend the theorem to currents with friction.

Following this, we extend the present analysis to nonretroflecting separating and meandering currents, like the Gulf Stream, in section 3. It is shown that by inte-

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grating the steady zonal momentum equation over a zonal line crossing the separating current system, a relation between the inertial coastal current and meandering current exists that does not depend on the details of the separation. On the other hand, several studies show that the meandering current momentum flux does depend on the details of separation, leading to an "information paradox." To investigate this further, the analytical solution provided by Moore and Niiler (1974) is studied and is shown to suffer from a momentum imbalance. From a basin-wide point of view, a steady state is only possible when the vorticity input by the wind in the basin interior on a streamline is dissipated on the same streamline. So each streamline has to move through a dissipative region (see e.g., Pedlosky 1996, and references therein). By simple scale arguments, it is shown that these dissipative regions have to be of basin size for realistic dissipation coefficients.

We thus conclude in section 4 that inertial, steady, separating currents connected to a nondissipative (e.g., linear Sverdrup) interior flow are impossible.

2. Retroflecting currents

Consider a steady current with a poleward momentum flux that flows along a north–south-oriented wall that curves westward. The current follows the coast until it separates and retroflects into the ocean interior. By retroflection we mean that the just-separated current makes an anticyclonic loop of more than 90°. To be more precise, we restrict retroflecting currents to currents where the cross-current-integrated momentum transport makes an anticyclonic loop of more than 90°.

Assume a reduced gravity, or 1.5-layer, configuration, with density difference $\Delta\rho$ and layer depth h. The results can be generalized by assuming that the upper layer has a finite depth h_0 outside the current, or to a barotropic description of the fluid. We assume the flow to be steady and show that this leads to a contradiction, by integrating the steady zonal momentum equation over a well-chosen area.

a. The frictionless current

Multiplying the steady zonal momentum equation by h and integrating over a region inside some contour ϕ gives

$$\int\int \left(huu_x + hvu_y - fhv + g'hh_x\right) dx\, dy = 0, \qquad (1)$$

in which g' is the reduced gravitational acceleration given by $g' = g\Delta\rho/\rho$. (For a reduced-gravity model in which the layer depth outside the current is h_0, the equation remains the same when $h = h_0 + \eta$. For a barotropic model g' is replaced by g, and η denotes the sea surface elevation.) The steady continuity equation reads

$$(hu)_x + (hv)_y = 0, \qquad (2)$$

so a streamfunction ψ can be defined by

$$\psi_x = hv\; \psi_y = -hu. \qquad (3)$$

Using continuity in Eq. (1) gives

$$\int\int \left[(hu^2)_x + (hvu)_y - f\psi_x + \frac{1}{2}g'(\eta^2)_x + g'h_0\eta_x\right] dx\, dy = 0. \qquad (4)$$

When the active layer outcrops in the reduced gravity model, $h_0 = 0$. Stokes's theorem can be used to find

$$\int_\phi huv\, dx - \int_\phi \left(hu^2 - f\psi + \frac{1}{2}g'\eta^2 + g'h_0\eta\right) dy = 0. \qquad (5)$$

Note that by this choice of signs in front of the integrals, we study the zonal momentum flux into the region of integration. This equation was also derived by NP for the case $h_0 = 0$, that is, when the upper-layer outcrops. The new Eq. (5) also applies when the upper-layer thickness is h_0 outside the current and for the barotropic case with g' replaced by g. NP derive a contradiction from this equation, leading to their momentum imbalance paradox. To derive that contradiction, NP assume a special condition on the fluid flow out of the integration area. This condition provides a strong limitation on the generality of their results, as is discussed in the appendix.

One might expect that the occurrence of such a contradiction depends on the direction of the momentum flux out of the area, because that direction determines the sign of the zonal momentum flux out of the area in the momentum budget. In Fig. 1 three directions for the momentum flux out of the area are indicated, which cover all possible configurations. The outward momentum flux can be eastward, southeastward, southward, or southwestward, but the analysis is the same for southward and southeastward momentum flux. In Figs. 2–4 three possible realizations of the momentum flux directions corresponding to the three situations in Fig. 1 are given, together with three contours ϕ that define the areas over which the momentum equation is integrated. (Note that we do not have to assume that mass and momentum transport are in the same direction.) In the following, these three cases are treated separately. For

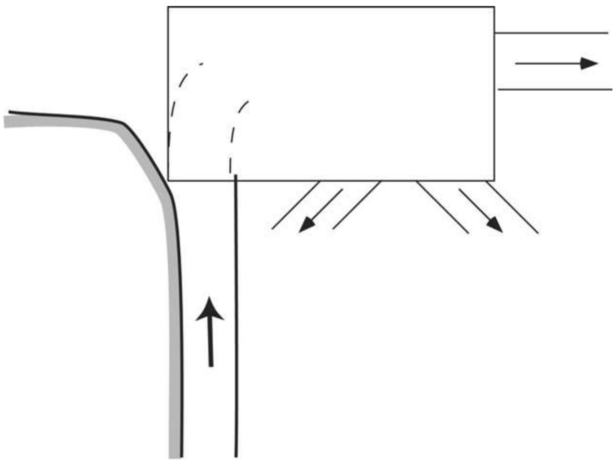

FIG. 1. The three possible configurations for the current after retroflection: eastward momentum transport, southeastward momentum transport, and southwestward momentum transport.

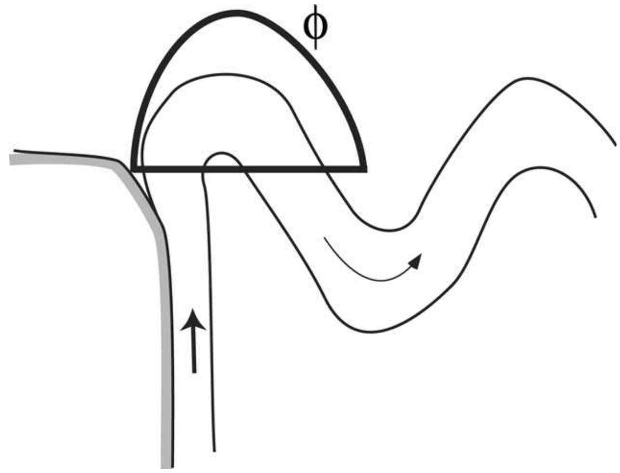

FIG. 2. Integration area for the zonal momentum balance for a southeastward momentum transport of the current after retroflection. The thick line denotes the integration area and the thin lines denote possible inner and outer streamlines of the flow.

clarity, we treat the frictionless case first in the next section and discuss the role of friction later. Also, we put $h_0 = 0$, but, as the reader can easily verify, the analysis below holds also when $h_0 \neq 0$ and for the barotropic case.

1) SOUTHWARD AND SOUTHEASTWARD OUTFLOW

We first deal with currents that make a strong turn southeastward and meander relatively mildly further on, as depicted in Fig. 2. Or rather, the cross-current-integrated momentum flow has this configuration as explained earlier. The integration contour ϕ runs straight east through the current, bends northward, and closes back on itself near the separation point.

For this configuration we find from (5), choosing $h = 0$ and $\psi = 0$ north of the current,

$$\int_{\phi} huv\, dx = 0. \tag{6}$$

For both the zonally integrated inward momentum flux in the northwestward direction and the zonally integrated outward momentum flux in the southeastward direction, the term on the left-hand side is negative and a contradiction arises. The meaning of the contradiction is that the chosen steady flow configuration cannot occur in reality. This does not come as a surprise, because the integral over the northward flow denotes the meridional advection of westward zonal momentum into the domain. The integral over the return flow denotes the meridional advection of eastward zonal momentum out of the domain. This situation can only persist when a source of eastward momentum is present in the domain, but that is not the case. This example illustrates the essence of the idea of NP that is also used here: integrate the zonal momentum equation for a steady reflecting current and derive a contradiction. Before we enter into the consequences of this contradiction, we study the other two cases.

2) EASTWARD OUTFLOW

We now discuss the case in which the current (or rather, the cross-current-integrated momentum trans-

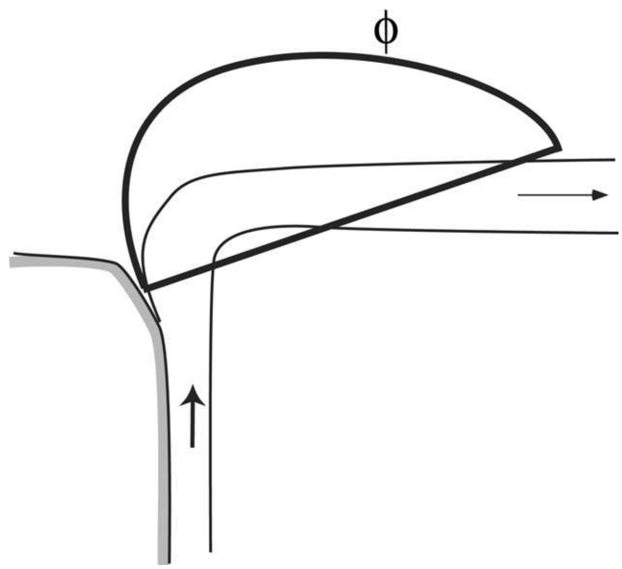

FIG. 3. Integration area for the zonal momentum balance for an eastward momentum transport of the current after retroflection. The thick line denotes the integration area and the thin lines denote possible inner and outer streamlines of the flow.

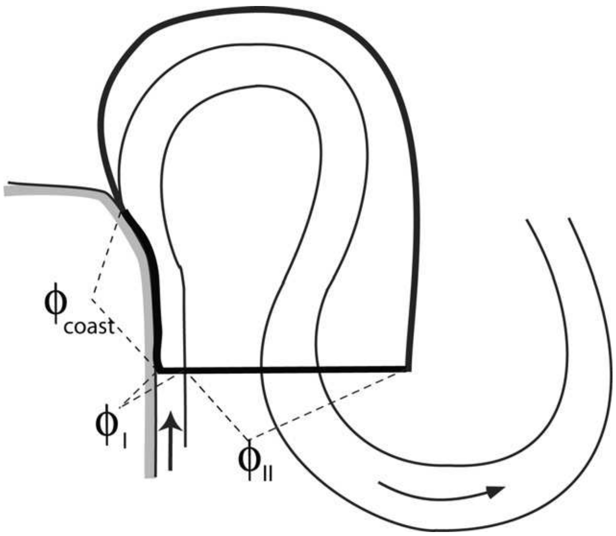

FIG. 4. Integration area for the zonal momentum balance for a southwestward momentum transport of the current after retroflection. The thick line denotes the integration area and the thin lines denote possible inner and outer streamlines of the flow.

port) does not meander after retroflection. The integration contour runs northeastward through the current and closes back on itself via a northward loop. The situation depicted in Fig. 3 can easily be evaluated by rotating the coordinate system over an angle θ such that the first situation arises again, with $0 < \theta < \pi/2$. Define an s coordinate northeastward, with velocity component p, and a τ coordinate perpendicular to s in the northwest direction, with velocity component q. We use $-f\psi_s = -(f\psi)_s + \psi f_s = -(f\psi)_s + \psi\beta\sin\theta$, in which β is the meridional derivative of the Coriolis parameter, as usual. Integrating the resulting momentum equation in the s direction leads to

$$\int_\phi hpq\,ds - \int_\phi \left(hp^2 - f\psi + \frac{1}{2}g'h^2\right) d\tau - \int\int \psi\beta\sin\theta\,ds\,d\tau = 0. \tag{7}$$

The second term is zero because that part of the contour runs outside the flow. Also, here we see that the first term is negative as is a new third term, leading again to a contradiction. The meaning of this contradiction is that this flow configuration cannot exist in steady state.

This result can easily be understood by realizing that the advective terms act the same as in the previous case. The β term is related to the tilted coordinate system. The mass transport in the positive τ direction leads to a Coriolis force directed to positive s (in the Northern Hemisphere). The mass transport in the negative τ direction induces a Coriolis force in the negative s direction. Now, from continuity, these two mass transports are equal, but due to the meridional variation of the Coriolis parameter (β effect), the latter force is larger. The resulting force thus implies a momentum flux in the negative s direction. Clearly, also this term needs to be compensated by a source of momentum in the positive s direction (''eastward''), but such a source is absent. So, the contradiction is also present in this case. This result does not contradict the steady numerical solution obtained by Arruda et al. (2004), because these authors treat the case in which the momentum transport makes a turn of only 90°, such that the zonal momentum balance connects to the coast.

3) Southwestward outflow

Now we treat the case shown in Fig. 4. For this case the imbalance is related to the connection of the flow to the coast. We choose the contour to run southward along the coast from the separation point, cross the current twice running eastward, and close the loop via a northward curve outside the current.

Let us first consider the case in which the momentum of the inflow is purely meridional. The integrated zonal momentum balance now reads

$$\int_{\phi_I} huv\,dx + \int_{\phi_{II}} huv\,dx + \int_{\phi_{coast}} huv\,dx - \int_{\phi_{coast}} \left(hu^2 - f\psi + \frac{1}{2}g'h^2\right) dy, \tag{8}$$

in which ϕ_I and ϕ_{II} are contours across the inflowing and outflowing parts of the flow (see Fig. 4). The first term in (8) is zero by our condition of purely meridional inflow. The second term is zero, too: we can always shift the contour in the meridional direction such that this term is zero, because north of this line the momentum flux is southwestward by definition, while south of it the momentum flux has to turn southeastward. Because the coast is a streamline, the two advective terms and the streamfunction term vanish there, and only the pressure term remains. So, the momentum balance reduces to

$$-\int_{\phi_{coast}} \frac{1}{2}g'h^2\,dy = 0. \tag{9}$$

This condition cannot be met by any flow. The physical explanation is that no zonal momentum enters the domain with the current, but the coast puts eastward momentum into the domain, without compensation. This result again means that at least one of our assumptions regarding the flow characteristics has to be relaxed.

Now we relax the condition of pure meridional momentum inflow and rotate our coordinate system anticlockwise over angle θ to align it with the southeastern boundary of the fluid domain. The coordinates are chosen similar to the previous case of eastward momentum outflow. The integrated momentum balance in the s direction now becomes

$$\begin{aligned}&\int_{\phi_{\mathrm{I}}} hpq\,ds + \int_{\phi_{\mathrm{II}}} hpq\,ds + \int_{\phi_{\mathrm{coast}}} hpq\,ds\\ &\quad - \int_{\phi_{\mathrm{coast}}} \left(hp^2 - f\psi + \frac{1}{2}g'h^2\right)\,d\tau\\ &\quad - \int\!\!\int \psi\beta\sin\theta\,ds\,d\tau = 0,\end{aligned} \tag{10}$$

in which ϕ_{I} and ϕ_{II} are contours as before (see Fig. 4, but now rotated over an angle θ). We now choose the coordinate system such that

$$\int_{\phi_{\mathrm{I}}} hpq\,ds + \int_{\phi_{\mathrm{II}}} hpq\,ds \approx 0. \tag{11}$$

This can always be done because the first integral is relatively small as the flow will tend to follow the coast, while the sign and magnitude of the second integral can be changed by a slight meridional displacement of the integration contour. Again, the advection terms along the coast cancel, and the streamfunction term is zero there, too. So, we are left with

$$-\int_{\phi_{\mathrm{coast}}} \frac{1}{2}g'h^2\,d\tau - \int\!\!\int \psi\beta\sin\theta\;ds\,d\tau = 0. \tag{12}$$

The first term is positive because the integration is performed in the negative τ direction, and the second term is negative. To study a possible momentum imbalance we determine the order of magnitude of the different terms. For the pressure term, we approximate the average layer depth on the integration path along the coast as $H/3$. [This corresponds with the value of Moore and Niiler (1974), some 200 km from the separation point.] For the β term, geostrophic velocities are used as first-order estimates. The order of magnitude of the two terms then becomes

$$\frac{g'H^2L_\tau}{18}\;\frac{g'H^2\beta L_\tau M\sin\theta}{8f_0}, \tag{13}$$

in which L_τ is the stretch of coast from inflow to separation, and M is the width of the first meander. The meander takes up about half of the rectangle $L_\tau \times M$, and the maximum streamfunction is about $1/2g'H^2/f_0$, adding an extra factor 1/4 in the last term. The order of magnitude of the β term to the pressure term is

$$\frac{2M\beta\sin\theta}{f_0}. \tag{14}$$

Due to geometric constraints θ cannot be much larger than 20° to prevent self crossing of the meandering flow (see Ou and de Ruijter 1986). Let us use $\sin\theta \le 0.3$. Using as typical midlatitude values $M = 400$ km, $\beta = 2\ 10^{-11}\ \mathrm{m}^{-1}\ \mathrm{s}^{-1}$, and $f_0 = 10^{-4}\ \mathrm{s}^{-1}$, we find that this ratio is about 0.05, so that again an imbalance exists.

Hence, this flow configuration cannot be steady. A possible way out might be to impose a recirculation cell within the first meander. This would have two direct effects: the maximum value of ψ would increase, but the average value of ψ over the integration area would grow much less because the area over which ψ is maximal decreases. A second effect is that the width of the meander M increases. However, even a factor of 2 increase in transport in the recirculation cell and a factor 2 increase in the width of the meander cannot negate our conclusion that the flow configuration cannot be steady.

A flow closer to the equator needs extra attention because the ratio between the two integrals becomes one when $f_0 \approx M\beta$, so at a distance of about $0.6M \approx 250$ km from the equator (using the equatorial β plane). At this distance from the equator, however, the separated current would reach the equator in a southward meander, which is unrealistic. Thus the imbalance remains.

b. The role of friction in retroflecting separating currents

The foregoing discussion of all three cases can be summarized as follows: a steady retroflecting current in which the cross-current-integrated momentum flux turns anticyclonically more than 90° in a frictionless reduced gravity (or barotropic) context cannot exist. Friction does not resolve the momentum imbalance because the momentum fluxes now comprise both advective and diffusive terms, and the reasoning used above can be used on these total fluxes, too. This can be understood by realizing that we do not solve the equations of motion including friction but impose a (quite general) geometry on the flow and then derive a contradiction. Friction can change the direction of the outflow, but all possible outflow directions have been treated in the three cases above. Friction might prevent the current from retroflecting, but that falls outside the scope of this section.

As an example, we treat the southward and southeastward outflow case here. Frictional terms appear at the southern boundary of the integration domain as depicted in Fig. 2. The zonal momentum balance now becomes

$$\int_{\phi} huv + Ahu_y \, dx = 0. \tag{15}$$

The paradox lies in the fact that by definition in this case the *total* zonal momentum of the influx is northwestward, giving a negative contribution to the integral, and the *total* zonal momentum of the outflow is southward or southeastward, giving again a negative contribution. Total zonal momentum fluxes that do not fulfill this case belong either to the eastward or to the southwestward outflow case, and all cases are treated.

Southwestward outflow needs a bit more attention, related to the fact that the imbalance was derived from an order of magnitude argument. In this case, the integrated momentum balance reads

$$\int_{\phi_\mathrm{I}} hpq\, ds + \int_{\phi_\mathrm{II}} hpq\, ds - \int_{\phi_\mathrm{coast}} \frac{1}{2} g' h^2 \, d\tau - \int\int \psi\beta \sin\theta \, ds\, d\tau - \int_{\phi_\mathrm{I}+\phi_\mathrm{II}} Ahq_\tau \, ds - \int_{\phi_\mathrm{coast}} Ahq_\tau \, ds + \int_{\phi_\mathrm{coast}} Ahq_s \, d\tau = 0. \tag{16}$$

The order of magnitude of these three new terms is

$$\frac{AHR_d^2 f_0}{L_\tau} \; \frac{AHR_d^2 f_0}{L_\tau} \; AHf_0 L_\tau, \tag{17}$$

of which the latter term is the largest. Its ratio to the coastal pressure term is $18A/(R_d^2 f_0)$. For a typical value of $A = 100\ \mathrm{m}^2\ \mathrm{s}^{-1}$, and a Rossby deformation radius $R_d = 30$ km, we find 0.01 for this ratio. We conclude that diffusion cannot balance the pressure term, and the momentum imbalance remains.

3. Nonretroflecting separating currents

In the following discussion of the steadiness of nonretroflecting inertial separating currents, two viewpoints are discussed. In the first, the zonal momentum equation is integrated over a zonal line connecting coastal current and separated current, and a relation is found between the zonal momentum fluxes in these two currents (see Fig. 5):

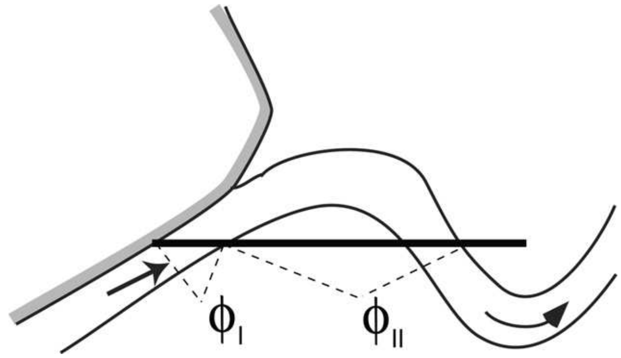

FIG. 5. Line of integration for a separating and meandering current. The thick line denotes the line of integration and the thin lines denote possible inner and outer streamlines of the flow.

$$hu^2(0) + \frac{1}{2} g' h^2(0) - \int_{\phi_I} (huv)_y \, dx - \int_{\phi_{II}} (huv)_y \, dx = 0. \tag{18}$$

The first three terms are related to the coastal current, and the other term is related to the separated meandering current and possible recirculations inside the first poleward meander that cross the line of integration. This equation sets a relation between the zonal momentum flux in the meander and the zonal momentum flux in the coastal current, which does not depend on details of the separation process. But the momentum structure of the meandering jet does depend on, for example, the angle of separation, which in turn depends on the coastal current structure and the curvature of the coastline, as shown by Ou and de Ruijter (1986). So given a steady coastal current, no a priori reason exists why the momentum flux in the separated meandering current is such that (18) is fulfilled. Obviously, this is related to the hyperbolic nature of the momentum equations.

This argument is based on a local view of the separation process, and in a basin-wide view one can argue that the flow is forced to fulfill the steady zonal momentum balance, and we cannot choose the momentum structure of the coastal current at will. We present a discussion of the local viewpoint first, and then add a short discussion on this basin-wide viewpoint.

a. The local viewpoint

To illustrate the local viewpoint, we consider the situation in which the meandering jet is approximated by a thin jet. In the thin-jet approximation, variations along the jet axis are assumed small in comparison with variations normal to the jet axis. It has been used extensively to study the stability and evolution of meandering currents because it decouples the along-track and the cross-track problems (e.g., Warren 1963; Robinson and Niiler 1967; Robinson et al. 1975; Flierl and Robinson 1984; Pratt 1988; Cushman-Roisin et al. 1993).

To find an analytical expression for the last term in (18), we introduce the stream-following coordinates s and r, in which s increases along the streamline and r is perpendicular to it pointing to the right, with $r = 0$ at the free streamline. These are related to our x, y frame as $dx = \sin\theta\, dr + \cos\theta\, ds$ and $dy = -\cos\theta\, dr + \sin\theta\, ds$ in which θ is the angle between the direction of the jet axis and the zonal direction (see, e.g., Robinson and Niiler 1967, p. 272). The y derivative becomes

$$\left(\frac{\partial}{\partial y}\right)_x = \left(\frac{\partial s}{\partial y}\right)_x \frac{\partial}{\partial s} + \left(\frac{\partial r}{\partial y}\right)_x \frac{\partial}{\partial r} = \sin\theta \frac{\partial}{\partial s} - \cos\frac{\partial}{\partial r}, \tag{19}$$

such that

$$\begin{aligned}\int_{\phi_{II}} (huv)_y\, dx &= \int_{\phi_{II}} \sin\theta (huv)_s - \cos\theta (huv)_r\, dx,\\ &= \int_{\phi_{II}} (hp^2 \sin\theta\, \cos\theta)_s \sin\theta - (hp^2 \sin\theta \cos\theta)_r \cos\theta\, dx,\\ &= \int_{\phi_{II}} hp^2 \cos 2\theta \frac{d\theta}{ds} \sin\theta - (hp^2)_r \sin\theta \cos^2\theta\, dx,\end{aligned} \tag{20}$$

in which p is the streamwise velocity in the s direction. In the thin-jet approximation, both θ and $d\theta/ds$ are functions of y only (see below). In that case, all goniometric terms can be taken out of the integrals, and we can use that along the x axis $dy = 0$, so that $\sin\theta ds = \cos\theta dr$, and hence

$$dx = \sin\theta dr + \frac{\cos^2\theta}{\sin\theta} dr = \frac{dr}{\sin\theta}, \tag{21}$$

leading to

$$\begin{aligned}\int_{\phi_{\rm II}} (huv)_y\, dx &= \cos 2\theta \frac{d\theta}{ds} \int_{\phi_{\rm II}} hp^2\, dr - \cos^2\theta \left[hp^2\right]^{r=0}_{\rm interior}\\ &= \cos 2\theta \frac{d\theta}{ds} \int_{\phi_{\rm II}} hp^2\, dr,\end{aligned} \tag{22}$$

in which we used $h(r = 0) = 0$ and $p_{\rm interior} = 0$. The curvature of the flow is defined as the inverse of the radius of curvature, so

$$\kappa = -\frac{d\theta}{ds} = -\frac{d\theta}{dy} \sin\theta = \frac{d\cos\theta}{dy}, \tag{23}$$

in which the second equality follows from $dr = 0$. In the thin-jet approximation, κ is found as

$$\kappa = -\alpha(y - y_c), \tag{24}$$

in which α is a constant dependent on details of the flow field and y_c is the central latitude of the jet. This expression is related to the fact that eastward advection by the jet is balanced by westward motion due to relative vorticity advection and the β effect (see, e.g., Robinson and Niiler 1967; Moore and Niiler 1974; Ou and de Ruijter 1986). This latter equation allows us to evaluate the y dependence of $\cos\theta$, as

$$\cos\theta = \int_{y_c}^{y} \kappa\, dy + \cos\theta_c = -\frac{\alpha}{2}(y - y_c)^2 + \cos\theta_c, \tag{25}$$

in which θ_c is the angle of the jet axis at the central latitude. As shown by Ou and de Ruijter (1986), θ_c is the angle of the flow at the separation point. When $\theta_c =$ 90° the jet is pointing meridionally at its midaxis, and when $\theta_c = 0°$ the jet points along the x axis and no meanders are present. The zonally integrated momentum balance (18) becomes the following for a thin jet:

$$\begin{aligned}hu^2(0) + \frac{1}{2}g'h^2(0) - \int_{\phi_I} (huv)_y\, dx\\ + \kappa \cos 2\theta \int_{\phi_{II}} hp^2\, dr = 0,\end{aligned} \tag{26}$$

with expressions for κ and $\cos\theta$ given in (24) and (25).

Moore and Niiler (1974) obtain an analytical solution for the complete circulation in an ocean basin, including the separation process. They assume that the separating current can be approximated by a thin jet, as used above. For their northward coastal current, we find

$$\frac{1}{2}g'h^2(0) = \frac{g' f_0 \beta}{2P_0^2}(y_c - y), \tag{27}$$

in which P_0 is the potential vorticity of the flow, assumed uniform. Because the zonal velocity is zero in the coastal current to first order, the first three terms in (26) reduce to the expression above. Very close to the separation point $h \downarrow 0$, the zonal velocity becomes important, so that the third term takes over. But we stay away from that point.

In the thin jet, they find

$$p = \sqrt{\frac{g'f_c}{P_0}} \exp\left(-\frac{f_c P_0}{g'} r\right)$$
$$h = \frac{f_c}{P_0}\left[1 - \exp\left(-\frac{f_c P_0}{g'} r\right)\right]. \tag{28}$$

In the Moore and Niiler (1974) solution, the factor α in the curvature κ is given by

$$\alpha = 3\beta\sqrt{\frac{P_0}{g'f_0}}. \tag{29}$$

We then find from (26)

$$\int_{\phi_{\mathrm{II}}} (huv)_y \, dx = -\frac{\kappa g' f_c}{6P_0^2} \cos 2\theta \sqrt{\frac{g'f_c}{P_0}}. \tag{30}$$

Using the relation between κ and y and $\cos\theta$ and y from (24) and (25), we find for their solution

$$\int_{\phi_{\mathrm{II}}} (huv)_y \, dx = \frac{g'f_c\beta}{2P_0^2}(y_c - y)\left[1 - \frac{9\beta^2 P_0}{2g'f_c}(y_c - y)^4\right]. \tag{31}$$

(Note that $\theta_c = 90°$ in their case.) When $y \approx y_c$ the momentum balance over the line of integration is fulfilled because both (27) and (31) vanish. However, away from y_c, so along a latitude line farther south, the two equations do not add up to zero, also not approximately because the last term in the square brackets in (31) is not small but of order one as it arises from the $\cos^2\theta$ term. We thus find that the solution of Moore and Niiler (1974) does not fulfill the zonally integrated zonal momentum balance. The reason for this is that the very complicated flow at separation (see Moore and Niiler 1974) is not matched to the thin-jet solution. The solutions for the separating current and their thin-jet solution are presented independently by Moore and Niiler, and no connection is made between these two solutions. Arruda et al. (2004) show a steady numerical solution of a jet separating at 90°. However, the flow is highly viscous with a horizontal viscosity of 1500 $\mathrm{m^2\,s^{-1}}$. Such a high viscosity can only come about as effective viscosity due to eddies, but these are not present in a steady state. Consistent with what we find here, when the authors lower the viscosity to more realistic values, the solution becomes unsteady.

This suggests that a steady flow in this configuration is perhaps impossible. To illustrate this further, we keep the inertial coastal current at the line of integration the same as Moore and Niiler (1974), but we change the stretch of coast between the line of integration and the separation point such that $\theta_c \neq 90°$. This leads to either a retroflecting current or a meandering flow similar to the Gulf Stream. Over the line of integration, the contribution from the coastal current does not change, but the first meander contribution becomes

$$\int_{\phi_{\mathrm{II}}} (huv)_y \, dx = \frac{g'f_c\beta}{2P_0^2}(y_c - y)\left[1 - 2\left(\cos\theta_c - \frac{3\beta}{2}\sqrt{\frac{P_0}{g'f_c}}(y_c - y)^2\right)^2\right], \tag{32}$$

which again does lead to a momentum imbalance in (26). This illustrates even more emphatically that the zonal momentum flux term of the separated current depends on the coastline position and curvature just before the separation point (see also Ou and de Ruijter 1986). The point is that given a coastal current, a steady separation is unlikely because the first meander has a momentum structure related to details of the separation process, while at the same time that momentum structure is prescribed by the coastal current before the actual separation through the zonally integrated zonal momentum balance. This situation could be termed the *information paradox*.

b. The basin-wide view

One can take the stance that the arguments presented above are open for discussion because when a steady flow is *imposed* on a separating current, the momentum structure in the coastal current and meanders have to adjust such that the zonal momentum balance is fulfilled. Numerical simulations of (statistical) steady circulations suggest that a steady separation can be achieved by either a strong recirculation cell in the northwest corner of the domain or by an overshoot of the current, forming a long loop current in which both inertia and dissipation are of order one (see, e.g., Cessi et al. 1990; Pedlosky 1996, and references therein). The analysis of both situations has been done mostly using Stewart's constraint, which says that vorticity input on a latitude circle by the wind over the interior of the basin has to be balanced by vorticity dissipation *at the same latitude* (Stewart 1964). This constraint rests on the assumption that the relative vorticity in the western boundary current can be approximated by the zonal derivative of the meridional flow (i.e., $\zeta \approx v_x$). However, in an inertial recirculation cell, this is questionable, and we will not use this constraint below.

In the following scaling analysis we show that a steady solution to the separation problem is not possible when a linear Sverdrup-like interior flow exists over the majority of the basin and inertia dominates in the western boundary current. [Numerical steady solutions show that the interior circulation becomes nonlinear, too, in the inertial case (Sheremet et al. 1997; V. Palastanga et al. 2007, personal communication), but that seems to contradict current observations.]

The vorticity input by the wind on each streamline that runs through the interior has to be dissipated in the separation area, where the excess relative vorticity of the flow has to be dissipated to allow a transition to the Sverdrup interior:

$$\int_{\text{separation}} A\Delta\zeta\, ds \approx -\int_{\text{Interior}} \frac{\tau_y^{(x)}}{\rho_0 H} ds, \tag{33}$$

in which the integral runs over a streamline. Locally, in the separation region, dissipation is balanced by advection of total vorticity:

$$A\Delta\zeta \approx v\zeta_y + \beta v. \tag{34}$$

The scaling argument given below is based on these two relations. The order of magnitude of the length of the separation area including the recirculation cell and/or an overshooting loop is given by l_s, and the width of the area in which the vorticity is dissipated in the separation area by l_d. We assume that the majority of the transport runs through this dissipative region in the separation area, so that in the order of magnitude of the streamfunction there is the interior Sverdrup value ψ_I. The first relation (33) then leads to

$$l_s A \frac{\psi_I}{l_d^4} \approx \beta\psi_I \quad \text{or} \quad l_s \approx \frac{\beta l_d^4}{A}. \tag{35}$$

To evaluate the second condition, we first notice that the planetary advection term is at most as large as the relative vorticity advection, because their ratio is

$$\frac{\beta v}{v\zeta_y} \approx \frac{\beta l_d^2 l_s}{\psi_I}. \tag{36}$$

To evaluate this further we can introduce the inertial length scale based on the Sverdrup transport: $\delta_I = \sqrt{\psi_I/\beta L}$, in which L is the length scale of the meridional size of the basin. The ratio becomes

$$\frac{\beta v}{v\zeta_y} \approx \frac{\beta l_d^2 l_s}{\beta \delta_I^2 L} \approx \left(\frac{l_d}{\delta_I}\right)^2 \frac{l_s}{L}. \tag{37}$$

When the western boundary current is mainly inertial, the frictional length scale l_d will be at most as large as the inertial length scale. Furthermore, $l_s \ll L$ for a recirculation cell, and $l_s \leq L$ for the overshooting loop. Hence, the planetary vorticity advection will be at most of the order of magnitude of the relative vorticity advection, and the local vorticity balance in the dissipative region of the separation area relates vorticity dissipation to relative vorticity advection, leading to

$$A\frac{\psi_I}{l_d^4} \approx \frac{\psi_I^2}{l_d^3 l_s} \quad \text{or} \quad l_s \approx \frac{\psi_I}{A} l_d. \tag{38}$$

Combining this with the estimate found from the dissipation of the vorticity input of the wind, we find, again using the inertial length scale defined above,

$$l_s \approx \left(\frac{\delta_I}{\delta_d}\right)^3 \left(\frac{L}{\delta_I}\right)^{1/3} L, \tag{39}$$

in which we introduced a dissipative length scale (from a balance between βv and local dissipation) $\delta_d = (A/\beta)^{1/3}$. When $\delta_d < \delta_I$ the length scale of the separation area is much larger than the meridional size of the basin, which is inconsistent with our assumption that the interior flow is Sverdrupian. This would be the case if we consider A to be the result of small-scale turbulent processes leading to a Munk-like dissipative layer, in which δ_d is the Munk scale. If, on the other hand, A contains contributions from the mesoscale eddies in the separation area, it can be much larger locally in the separation area. In that case, δ_d can be much larger than δ_I, and the length scale of the separation area l_s can be much smaller than that of the basin L. Hence, a Sverdrup interior flow with a strong steady inertial jet at the western boundary cannot balance the vorticity input by the wind, and eddies have to be present to induce extra dissipation.

We have to conclude, then, that even from a basin-wide viewpoint, steady separation is unlikely because it seems to be impossible to close the global vorticity balance in steady state, while keeping both a realistic inertial western boundary current and a realistic interior circulation.

4. Conclusions and discussion

In this paper we have confirmed that a steady separating and retroflecting current in a reduced gravity or a barotropic model suffers from a momentum imbalance paradox. It can be proven directly that such a current cannot fulfill the zonal momentum equation integrated over a suitably chosen area. We have shown that even friction cannot prevent this momentum imbalance. The apparent contradiction between previous work on separating currents, and the more recent work by NP is solved.

We also showed that nonretroflecting separating currents are likely to suffer from a similar momentum imbalance because the zonal momentum flux in the coastal flow is directly related to that in the meandering jet, while the former is not dependent on the details of the separation process, but the latter is to a great extend. This information paradox can give rise to a momentum imbalance in that given a coastal current, the separation process is unlikely to be steady. This point was illustrated by studying the case in which the free meandering jet is approximated by a so-called thin jet. Moore and Niiler (1974) use this approximation in their analytical solution for a subtropic gyre. We found that a momentum imbalance is present in their solution, related to the matching of the just-separated flow with the free meandering thin jet.

This local viewpoint of the separation process is complemented with a basin-wide view, in which a scaling argument is used to show that the size of the area of vorticity dissipation has to be larger than the whole basin, which is inconsistent with observations that do show a Sverdrup-like interior flow.

These points lead to the important conjecture that a considerable part of the variability in the World Ocean might be due to the impossibility of a steady separation, not to an instability of a free jet.

A point in favor of the analysis presented here on retroflecting currents is the work by Dijkstra and de Ruijter (2001) and W. M. Schouten (2003, personal communication), who use continuation techniques to follow steady states of the Agulhas through parameter space. By decreasing the friction parameter by continuation, they find that the current at some point overshoots the African continent and flows all the way to South America, retroflects there, and connects back to the wind-driven gyre in the Indian Ocean: the flow never fully retroflects in the open ocean, in accordance with our conclusions.

NP point to the possibility of ring shedding to solve the momentum imbalance, but one could also imagine that the retroflection itself starts moving westward, to absorb the excess westward momentum. This is the only solution in a linear model. The Agulhas Current, for instance, does show these westward intrusions into the South Atlantic Ocean (see, e.g., Lutjeharms and Van Ballegooyen 1988; Schouten et al. 2002).

In a multilayer ocean, the excess momentum in the retroflecting layer(s) may be transported to deeper layers. A countercurrent in deeper layers might solve the imbalance via the pressure-gradient term. Boudra and de Ruijter (1986) reported in their simulations a momentum transfer from the upper to lower layer, but their flow field is time varying. From a local point of view, there seems to be no direct reason for the countercurrent to balance the momentum exactly. The basin-wide argument is based on vorticity dissipation arguments, which do not depend directly on lower layers, again pointing to the impossibility of steady, separating, inertial currents in ocean basins with a Sverdrup interior.

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APPENDIX

The Momentum Imbalance Theorem of Nof and Pichevin

The analysis presented here is the same for the two basic cases treated by NP, in which the retroflected current does or does not flow along a zonal coast. NP take one of the contours on the integration area along a meridian and assume that the outflowing current is purely zonal there, so $v = 0$ at that meridional section (see Fig. A1). Looking at their equations, they also assume $v_x = 0$, because their y-momentum equation at the section reads

$$-fu = -g'h_y, \tag{A1}$$

so purely geostrophic. This assumption of NP leads to (in our notation)

$$-\int_\phi huv\,dx + \int_\phi \left(hu^2 + \int_y^L \beta\psi\,dy'\right) dy = 0, \tag{A2}$$

in which L is the northern edge of the return flow. All terms are positive, so a contradiction arises, generating their momentum imbalance paradox.

The assumption in the NP imbalance of the geostrophic balance (A1) for the outflowing current has serious consequences. If that flow were only *approximately* geostrophic, their imbalance would not arise. When it is exactly geostrophic, either the flow has to fulfill $u^2 = g'h$ at outflow or all zonal derivatives of h, u, and v vanish. This is shown as follows: continuity reads

$$(hu)_x + (hv)_y = 0, \tag{A3}$$

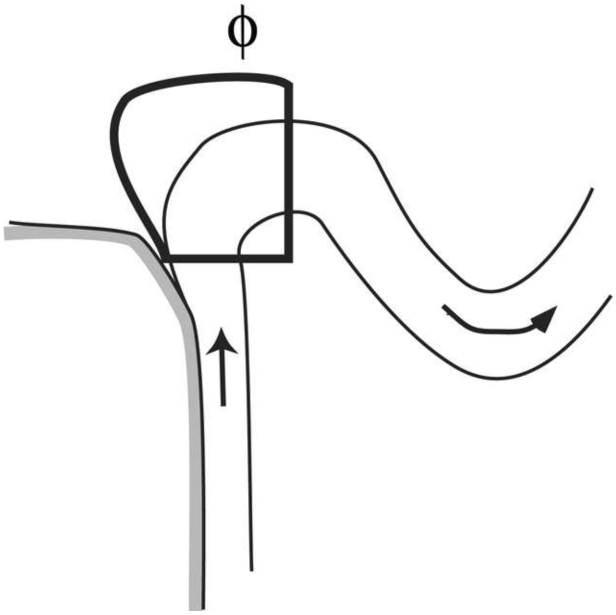

FIG. A1. Integration area typically used by NP. The thick line denotes the integration area and the thin lines denote possible inner and outer streamlines of the flow.

which can be written as

$$uh_x = -hu_x - (hv)_y. \qquad (A4)$$

Note that when $v\,(y) = 0$ for all y, then so is $v_y = 0$, and, in fact, all meridional derivatives of v vanish. From the momentum equation in the zonal direction, we find, using continuity,

$$(u^2 - g'h)u_x = -vuu_y + fuv + g(hv)_y. \qquad (A5)$$

The meridional momentum equation gives, after differentiation to x,

$$uv_{xx} = -u_x v_x - (vv_y)_x - fu_x - g'h_{xy}. \qquad (A6)$$

Given these three functional relations, it is easy to investigate the consequences of the assumptions put by NP on the outgoing flow. The $v(y) = 0$ (and so all meridional derivatives of v) and $v_x = 0$ (from the assumption of geostrophy) lead with (A5) to either $u^2 = g'h$ or $v_x = 0$. Pursuing the last condition, we find from (A4) that $h_x = 0$, while (A6) gives $v_{xx} = 0$. Taking now the x derivatives of (A4), (A5), and (A6) shows that $u_{xx} = 0$, $h_{xx} = 0$, and $v_{xxx} = 0$. This process can be repeated ad infinitum, and the consequences given above are proven.

When all zonal derivatives of u, v, and h are zero, a Taylor series expansion of each of these variables from any point in the domain with respect to the outflow position where $v = 0$ and $v_x = 0$ shows that all variables are independent of the x coordinate, which is inconsistent with a retroflecting current and any downstream meander of the current.

Hence, the outflow condition as used by NP leads to $u^2 = g'h$ at outflow. Combining this condition with geostrophy leads to a relative vorticity of $\zeta = ½f$ everywhere along the meridional and to unrealistically high velocities for reasonable current widths L. For example, $L = 100$ km gives $u = 5$ m s^{-1} at midlatitudes. Such a current might be possible when the retroflected current flows parallel to a wall. Nof (1978) found solutions of this shape but assumes small Froude numbers, so $u^2 + v^2 \ll g'h$. Garvine (1987) treated the supercritical case by integrating the solution along the characteristics. He found the formation of fronts and a current that has a variable thickness along the wall, violating the assumptions in the derivation by NP.

We conclude that the seemingly weak condition of geostrophy at outflow results in a rather extreme condition on the outflow structure, so the proof by NP is not complete.

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