to a (to be verified later), we can at leading order replace b with a and we find the required result for $b-a$. Putting in numbers (Ω by definition equals $2\pi\,\text{day}^{-1}$), we find that $q = 1/582$. This verifies that b is very close to a, so the earlier approximation is justified.

It also shows that this simple model is off by a factor of about 2, because we did not take into account the effect of the oblateness on the geoid: in our approximation a surface point on the equator is in some sense above the spherical surface we assume for the gravitational part for the geopotential. Adding mass to fill up this extra height would reduce the geopotential of this point, so we need to move this point out to reset its geopotential to zero. In other words, the oblateness will move the $\phi = 0$ point on the equator outward compared to our simple model, corresponding to a larger q.

In fact this works quantitatively as well: a surface equator point lies a height qa above the surface, approximately corresponding to a geopotential, compared to the sphere, of $g_0 qa$. Filling up this extra height brings this point to the surface of the geoid and reduces the geopotential by approximately $g_0 qa$. To compensate for this reduction, the $\phi = 0$ point is located further outward by approximately qa. In other words, in reality we expect q to be about twice as big as in our simple model. So $b - a = qa$ with $q \approx \Omega^2 a/g_0$ (instead of $q = \Omega^2 a/2g_0$). Plugging in numbers we find $q \approx 1/291$, which is close to the observed result.

4.3 The adiabatic lapse Γ_a is the temperature change with height in a hydrostatic, isentropic fluid. From this definition one would expect that

► $$\Gamma_a = -\left(\frac{\partial T}{\partial p}\right)_s \frac{\mathrm{d}p}{\mathrm{d}z}.$$

Use the reciprocity relation for partial derivatives and a Maxwell relation to show that this is equivalent to Eq. 4.29.

Solution We can use the reciprocity relation to write

$$\left(\frac{\partial T}{\partial p}\right)_s = -\left(\frac{\partial T}{\partial s}\right)_p \left(\frac{\partial s}{\partial p}\right)_T = \left(\frac{\partial T}{\partial s}\right)_p \left(\frac{\partial v}{\partial T}\right)_p,$$

where in the last step we used the second Maxwell relation. Now writing hydrostatic balance as $\mathrm{d}p/\mathrm{d}z = -g/v$ we find

$$\Gamma_a = -\left(\frac{\partial T}{\partial p}\right)_s \frac{\mathrm{d}p}{\mathrm{d}z} = \frac{g}{v}\left(\frac{\partial v}{\partial T}\right)_p \left(\frac{\partial T}{\partial s}\right)_p = \frac{g\alpha_p T}{c_p},$$

as required. In the last step we used the definition of α_p and that $(\partial T/\partial s)_p = T/c_p$, by virtue of the reciprocal relation.

4.4 Assuming water has a constant heat capacity c and a constant thermal expansivity α, show that the temperature profile for an adiabatic lapse rate in water is

$$T = T_0 \exp(-g_0 \alpha Z/c).$$

The scale height for this profile is very large so a linear profile is appropriate for large height ranges. For water $\alpha \approx 2 \times 10^{-4}\mathrm{K}^{-1}$ and $c \approx 4.2 \times 10^3 \mathrm{J\,kg^{-1}\,K^{-1}}$ so that at typical temperatures the adiabatic lapse rate is less than about $0.15\,\mathrm{K\,km^{-1}}$.

SOLUTION By definition of the adiabatic lapse rate we have

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -\Gamma_a = \frac{g\alpha}{c} T,$$

with g, α and c assumed constant. The solution of this equation for T is the exponential profile proportional to $\exp(-Z/H)$ with scale height H,

$$H = \frac{c}{g_0 \alpha} \approx 2000\,\mathrm{km}$$

The lapse-rate for $|Z| \ll 2000\,\mathrm{km}$ is equal to T_0/H, or about $0.13\,\mathrm{K\,km^{-1}}$.

4.5 Estimate typical values for the geopotential thickness of the 1000 hPa–500 hPa layer when the surface temperature is 0°C and when the surface temperature is 20°C. By how much would the geopotential thickness of the 1000 hPa–500 hPa layer increase on a uniform increase in temperature of 1°C?

SOLUTION There are a few different ways to answer this question, and there is no single right answer. In the simplest approximation we can use the hypsometric equation for an isothermal ($T = T_0$) atmosphere to find

$$Z_1 - Z_0 = \frac{RT_0}{g_0} \ln(p_0/p_1).$$

In this approximation, the thickness of the 1000 hPa–500 hPa layer at $T_0 = 0°C$ is 554 dam and at $T_0 = 20°C$ we find a thickness of 594 dam. We should expect this to be an overestimate of the real thickness, because the layer-mean temperature in the hypsometric equation will be lower than the surface temperature. In the isothermal approximation, each degree Celsius increase in temperature corresponds to an increase in thickness of about 2 dam.

In a more accurate approximation we assume the temperature to decrease with height, for example with a fixed lapse rate Γ, so that $T = T_0 - \Gamma Z$. In this approximation we can write T as a function of p, using Eq. 4.19,

$$T = T_0\,(p/p_0)^{R\Gamma/g_0}.$$

With this approximation, we can integrate the hypsometric equation to find

$$Z_1 - Z_0 = \int_{p_1}^{p_0} \frac{RT_0}{g_0}\,(p/p_0)^{R\Gamma/g_0 - 1}\,\mathrm{d}p = \frac{T_0}{\Gamma}\left(1 - \left(\frac{p_1}{p_0}\right)^{R\Gamma/g_0}\right).$$

Taking $\Gamma = 6.5\,\mathrm{K\,km^{-1}}$ we find for the thickness of the 1000 hPa–500 hPa layer

$$(Z_1 - Z_0)\,(\mathrm{dam}) = 1.9 \times T_0(\mathrm{K}).$$

For $T_0 = 0°C$ we find a thickness of 519 dam, and for $T_0 = 20°C$ a thickness of 557 dam. Each degree Celsius increase in T_0 corresponds to a 1.9 dam increase in thickness.

The isothermal approximation can be improved, by using for T_0 an estimated temperature in the middle of the layer, instead of the surface temperature. Of course, we do not know where the middle of the layer is precisely, but the 750 hPa level in the standard atmosphere is about $Z_m = 2.5$ km high. So we could adjust the surface temperature T_0 to $T_0 - \Gamma Z_m$ and then use this temperature in the isothermal approximation. Using this method and again taking

$\Gamma = 6.5\,\mathrm{K\,km^{-1}}$ we find at a surface temperature of 0°C a thickness of 521 dam and for a surface temperature of 20°C a thickness of 562 dam.

4.6 Show that for an isothermal atmosphere the Brunt–Väisälä frequency is given by:

$$N^2 = \frac{g^2}{c_p T}.$$

The isothermal atmosphere is one of the most stable profiles observed in the atmosphere (for example, in the lower stratosphere). This provides a typical upper bound for the buoyancy frequency. Hence show that typical buoyancy periods are not shorter than about 5 minutes.

SOLUTION It is easiest to use the ideal gas approximation for Eq. 4.44, where we take $\Gamma_a = \Gamma_d = g/c_p$. We then get for the buoyancy frequency N in an ideal gas:

$$N^2 = \frac{g}{T}\left(\frac{g}{c_p} + \frac{\mathrm{d}T}{\mathrm{d}z}\right).$$

The required result now follows when putting in the isothermal condition $\mathrm{d}T/\mathrm{d}z = 0$. Putting in numbers (take $T_0 = 0°\mathrm{C}$) we get $N = 1.9 \times 10^{-2}\mathrm{s}^{-1}$. The corresponding period is $2\pi/N = 335\,\mathrm{s} = 5.6\,\mathrm{min}$. Expecting typical buoyancy frequencies to be less than the isothermal one, we expect typical periods to be not less than 5 minutes or so. Observed values of N are closer to about $N = 10^{-2}\mathrm{s}^{-1}$ in the mid-latitude troposphere, corresponding to a period of about 10 minutes.

4.7 Total air temperature. The air at the skin of an aircraft has the same speed as the aircraft itself because viscosity makes the air "stick" to the aircraft. In the frame of reference moving with the aircraft the air then has to decelerate from the aircraft speed to a standstill at the skin of the aircraft. Use the Bernoulli equation to show that the temperature T_t of this decelerated air is related to the actual air temperature T by

$$\frac{T_t}{T} = 1 + \frac{\gamma - 1}{2}\left(\frac{\mathscr{V}}{\mathscr{C}}\right)^2,$$

with $\gamma = c_p/c_v$, $\mathscr{V}$ the flight speed of the aircraft, and $\mathscr{C} = \sqrt{\gamma R T}$ the speed of sound. The ratio $\mathscr{V}/\mathscr{C}$ is called the *Mach number.* The temperature T_t of the decelerated air is called the *stagnation temperature* or the *total air temperature.* In this context, the actual air temperature T is also called the *static air temperature.* A probe on an aircraft measures the total air temperature and uses the flight speed to calculate the static air temperature.

Calculate typical values of the total air temperature for an aircraft at cruising altitude. Would an aircraft expand appreciably due to this heating effect? The difference between T_t and T can be quite large: the same effect is responsible for heating up spacecraft or meteorites when they (re-)enter the Earth's atmosphere. In popular literature it is often said that this heating is due to friction. This is a confusing way of describing the phenomenon. Friction only ensures that the air sticks to the space ship; the heating itself is due to the Bernoulli effect, or its supersonic equivalent (at very low densities ballistic effects of individual molecules need to be taken into account).

SOLUTION In the reference frame of the aircraft, that is, the reference frame where we do the measurements, the air on its skin has zero velocity but the air further ahead of the aircraft has a speed of $\mathscr{V}$ in the direction of the aircraft. So in this frame of reference the airspeed decelerates from $\mathscr{V}$ to zero. Because this deceleration occurs adiabatically, the Bernoulli function is conserved. For an ideal gas, we get

$$(c_p T + \phi)_{\text{ahead of aircraft}} + \frac{1}{2}\mathscr{V}^2 = (c_p T + \phi)_{\text{at skin of aircraft}},$$

where we used the ideal gas expression for the enthalpy (the zero point energy is irrelevant in this context), and we set $\mathscr{V} = 0$ at the skin of the aircraft. Because the geopotential is not expected to change much during the deceleration of the air, we get

$$c_p T + \frac{1}{2}\mathcal{V}^2 = c_p T_t$$

where we have defined the temperature ahead of the aircraft as the static air temperature, T, and the air temperature at the skin of the aircraft at the as the total air temperature T_t. This equation can be rearranged to

$$\frac{T_t}{T} = 1 + \frac{\mathcal{V}^2}{2c_p T} = 1 + \frac{\gamma R T}{2c_p T}\left(\frac{\mathcal{V}}{\mathcal{C}}\right)^2,$$

where we used $\mathcal{C}^2 = \gamma R T$. Now writing $R = c_p - c_v = c_p(1 - 1/\gamma)$ we find the desired result.

The typical cruise speed of a passenger aircraft may be about $\mathcal{V} = 900\,\mathrm{km\,hr^{-1}} = 250\,\mathrm{m\,s^{-1}}$. The soundspeed near the tropopause level (with $T \approx 220\,\mathrm{K}$) is about $\mathcal{C} = 300\,\mathrm{m\,s^{-1}}$. We then find

$$T_t/T \approx 1.14,$$

corresponding to a total air temperature of about $T_t \approx 250\,\mathrm{K}$, compared to a static air temperature of $T = 220\,\mathrm{K}$. The difference is about 30°C. With a linear thermal expansion coefficient of about $2 \times 10^{-5}\,\mathrm{K^{-1}}$ (we used an approximate value for aluminium), we find a fractional expansion of 6×10^{-4} of the length of the aircraft. At a typical length of perhaps 60 m, this corresponds to an additional length of nearly 4 cm.

4.8 Entropy in statistical mechanics. Using the definition, Eq. 4.67, of the probability P_i of a microstate i, show that the entropy of a system, Eq. 4.78, can be expressed as

► $$S_A = -k_B \sum_i P_i \ln P_i.$$

This expression for entropy is called the *Gibbs entropy.* Besides its central role in statistical mechanics, it is also the relevant expression of Shannon's information entropy, see footnote 14. Show that if the microstates i have an equal probability, the Gibbs entropy reduces to the Boltzmann entropy, Eq. 2.35.

SOLUTION Taking the logarithm of Eq. 4.67, we get

$$\ln P_i = -U_{A,i}/k_B T - \ln Z.$$

Multiply through by $-k_B P_i$ and sum over all microstates, to find

$$-k_B \sum_i P_i \ln P_i = \sum_i P_i U_{A,i}/T + k_B \ln Z \sum_i P_i = \langle U \rangle / T + k_B \ln Z,$$

where in the last step we used that the probabilities P_i are normalized, and thus add up to 1. Comparing this equation with the definition of S_A in Eq. 4.78, we find the desired result.

At equal probabilities for the microstates, we have $P_i = 1/\mathscr{W}$, with $\mathscr{W}$ the number of microstates. So we find for the Gibbs entropy

$$S_A = -k_B \sum_i P_i \ln P_i = k_B \sum_i (1/\mathscr{W}) \ln \mathscr{W} = k_B \ln \mathscr{W},$$

which is the Boltzmann expression for the entropy, Eq. 2.35.