Lower Boundary Condition
for the
Quasigeostrophic Equations in Isentropic Coordinates

MAARTEN AMBAUM*
Royal Netherlands Meteorological Institute, De Bilt, the Netherlands

ABSTRACT

The boundary conditions for the quasigeostrophic equations in isentropic coordinates are examined. It is shown that, if one prescribes the height of the lower boundary, the barotropic mode must have a finite effective Rossby radius. The resulting equation for this mode is the equivalent barotropic vorticity equation. The extension to multi-mode systems is treated, with the two-mode system as a specific example. This two-mode system is equivalent to a certain two-layer system. Furthermore, it is shown how weak diabatic effects may be included in this framework. This is illustrated with a simple model of thermal relaxation for the equivalent barotropic vorticity equation. The potential vorticity forcing is in this model proportional to the streamfunction.

1 Introduction

The starting point of this study is the short note by Berrisford *et al.* (1993) on quasi-geostrophic potential vorticity in isentropic coordinates. In this note it is shown how the introduction of isentropic coordinates leads to a conceptual simplification of the derivation of the quasigeostrophic equations, as a result of the strict interpretation of vertical velocities in terms of diabatic effects. In the present study we concentrate on the introduction of a lower boundary condition in this model. We will show that the usual neglect of the so-called "non-Doppler" term is not necessary. The inclusion of the non-Doppler term effectively represents the allowance of dynamical pressure variations at the lower boundary. These pressure variations are related to variations in the mass content of the air column above the surface. As such we allow for vortex stretching effects in the air column, which, in turn, can be translated into a finite effective Rossby radius for the barotropic mode. The resulting equation for this barotropic mode is the equivalent barotropic vorticity equation.

We will treat two extensions to the equivalent barotropic model. The first consists of the inclusion of higher baroclinic modes. As a specific example the two-mode system is

*M.H.P. Ambaum, `m.h.p.ambaum@reading.ac.uk`

presented in its general form and in a form where a background stratification, which is typical for the troposphere and the lower stratosphere, is specified. A second extension is the inclusion of weak diabatic effects in the model. These diabatic effects are assumed not to alter the background stratification. As a specific example, we show how a simple model for thermal relaxation can be formulated for the equivalent barotropic vorticity equation.

The plan of this article is as follows: In the next section we summarize the quasi-geostrophic equations in isentropic coordinates. Section 3 is devoted to the treatment of the lower boundary condition. Here we also show how we may render the boundary conditions homogeneous, by introducing an appropriate potential vorticity sheet at the lower boundary. In section 4 these homogeneous boundary conditions are used to expand the equations into vertical modes. In section 5 we describe how the single-mode expansion leads to the equivalent barotropic model. Section 6 is devoted to the extension of the model to multiple modes, with the two-mode system as a specific example. In section 7 we show how we may include weak diabatic effects, illustrated with a thermal relaxation model for the equivalent barotropic vorticity equation.

2 The isentropic quasigeostrophic equations

In this section the quasigeostrophic model in isentropic coordinates, as described by Berrisford *et al.* (1993), is summarized. For details on the derivation of this model we refer to their note. For convenience we adopt the same notation.

All atmospheric variables are expressed as the sum of a barotropic (i.e. only dependent on potential temperature θ) reference value, denoted by subscript "0", and a small deviation from this reference value, denoted by a prime. For example, the pressure field p is written as

$$p(x, y, \theta, t) = p_0(\theta) + p'(x, y, \theta, t). \tag{1}$$

The geostrophic velocity $\mathbf{v}$ follows from the streamfunction ψ, which, in turn, is proportional to the deviation of the Montgomery potential M, by

$$\mathbf{v} = \mathbf{k} \times \nabla\psi = \frac{1}{f_0}\mathbf{k} \times \nabla M', \tag{2}$$

with $\mathbf{k}$ a unit vector pointing upward, and f_0 a reference value of the Coriolis parameter f. The horizontal derivatives ∇ are here, and throughout the rest of this article, to be taken with constant potential temperature. The Montgomery potential M is defined as

$$M = C_pT + gz, \tag{3}$$

with C_p the specific heat at constant pressure, T the temperature, g the gravitational acceleration, and z the height field.

The quasigeostrophic model in isentropic coordinates consists of the material conservation of quasigeostrophic potential vorticity q, following the geostrophic velocity. This is expressed as

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) q = 0. \tag{4}$$

The right-hand side of this equation is zero, which corresponds to frictionless and adiabatic conditions. A discussion of frictional and diabatic effects is deferred to section 7. The quasigeostrophic potential vorticity equals

$$q = f + \nabla^2\psi - \frac{f_0^2}{(dp_0/d\theta)}\frac{\partial}{\partial\theta}\left(\rho_0\theta\frac{\partial\psi}{\partial\theta}\right) = f + \mathcal{L}\psi, \tag{5}$$

where ρ is the density and where the linear differential operator $\mathcal{L}$ is introduced (note that $\mathcal{L}$ is elliptic because $(dp_0/d\theta) < 0$.) Equations 4 and 5 form a closed system in the prognostic variable q if the elliptic operator $\mathcal{L}$ can be inverted. This is the case if suitable boundary conditions on the streamfunction are supplied. This is the subject of the next section.

For our purposes we also need the following relations regarding the vertical structure of the atmosphere. The hydrostatic equation accurately describes the main vertical balance of the atmosphere. In isentropic coordinates it reads

$$\frac{\partial M}{\partial\theta} = \frac{C_pT}{\theta}. \tag{6}$$

The first law of thermodynamics can be written as

$$\frac{dQ}{T} = C_p\frac{dT}{T} - R\frac{dp}{p} = C_p\frac{d\theta}{\theta}, \tag{7}$$

where dQ is the heat input per unit of mass. Now combining the hydrostatic equation with the first law of thermodynamics we find that pressure deviations are related to deviations in the Montgomery potential, and consequently to the streamfunction, as

$$p' \approx \rho_0\theta\frac{\partial M'}{\partial\theta} = \rho_0 f_0\theta\frac{\partial\psi}{\partial\theta}. \tag{8}$$

This relation holds even in the presence of diabatic effects. It can be combined with the definition of the Montgomery potential and the ideal gas law to give the following relation between deviations in the height field and the streamfunction:

$$\frac{gz'}{f_0} = \psi - \theta\frac{\partial\psi}{\partial\theta}. \tag{9}$$

This relation states that the height deviation as a function of $p'/(\rho_0 f_0\theta)$ is the Legendre transform of the streamfuction as a function of θ

3 Boundary conditions

The elliptic operator $\mathcal{L}$ can be inverted if the streamfunction, its normal derivative or a combination of the two is provided at the boundaries. The horizontal boundary conditions depend on the horizontal geometry of the system. In a system with rigid walls, we should impose the no-flux condition, which means that no fluid penetrates the horizontal walls. In terms of the streamfunction this is equivalent to requiring ψ to be constant at the walls. Other possible geometries have no bounding walls, like for example a doubly periodic domain, a sphere or an infinite plane. In these geometries no horizontal boundary conditions are required, except for the infinite plane, where the energy should vanish at infinity.

The top boundary condition is fixed by requiring the pressure variations to vanish. Following Eq. 8, this implies the boundary condition

$$\left(\frac{\partial \psi}{\partial \theta}\right)_t = 0, \tag{10}$$

where the subscript "t" denotes evaluation at the top boundary level θ_t. For all practical purposes (excluding, for example, at the $p_0 = 0$ level of an isothermal atmosphere), this boundary condition can be applied at a finite θ_t. In this way we circumvent the problems of applying boundary conditions at infinite height, like treated in Pedlosky (1979) or Chapman and Lindzen (1970). Anyhow, the neglect of pressure variations above a certain isentropic level seems to be a rather unrestrictive condition, if this level is chosen high enough.

The lower boundary condition is fixed by the requirement that the height of the lowest boundary is prescribed. We now use Eq. 9 to relate the height of the lower boundary with the streamfunction. If we want to prescribe a fixed orographic profile $\eta(x, y)$ at the lower boundary, we have the following condition on the streamfunction at the lower boundary (denoted by subscript "s"):

$$\left(\frac{\partial \psi}{\partial \theta}\right)_s = \frac{\psi(\Theta_s)}{\Theta_s} - \frac{g\eta}{f_0 \Theta_s}, \tag{11}$$

where Θ_s is the potential temperature at the lower boundary. In accordance with quasi-geostrophic theory, η will be so small that Θ_s differs by a small amount from the fixed potential temperature θ_s at $z_0 = 0$. So we have

$$\Theta_s = \theta_s + \theta'_s, \qquad \text{where} \qquad \theta'_s \ll \theta_s. \tag{12}$$

We now linearize Eq. 11 around θ_s and obtain

$$\left(\frac{\partial \psi}{\partial \theta}\right)_s = \frac{\psi(\theta_s)}{\theta_s} - \frac{g\eta}{f_0 \theta_s}. \tag{13}$$

The physical picture of this boundary condition is that of an air parcel at the lower boundary which flows over a mountain while conserving its potential temperature. Note that we still allow for pressure variations at the lower boundary.

There are some features in which this treatment of the boundary conditions differs from that by Berrisford *et al.* (1993). First of all, we use a fixed lower boundary height as lower boundary condition whereas Berrisford *et al.* argue that height deviations are approximately advected over the lower boundary (here, "approximately" refers to the same approximation as in Eq. 12.) The latter is of course a rather strange boundary condition, because there is no reason to expect that the height of an air parcel on the lowest isentropic surface θ_s, that was originally located somewhat above the lower surface, may not change, even on a flat lower surface. In other words, we propose to replace the boundary condition $Dz'/Dt = 0$ by the stronger condition $z' = 0$ (for a lower surface with orography η we propose a replacement of the boundary condition $Dz'/Dt = -\mathbf{v} \cdot \nabla\eta$ by the stronger condition $z' = \eta$.) Secondly, we argue that that there is no reason to omit the contribution of ψ to the surface height field η in Eq. 13, which contribution leads to a so-called "non-Doppler term" in the lower boundary condition of Berrisford *et al.* We will show that this term gives rise to dynamical variations in the lower surface pressure (contrary the the passive pressure deviations as allowed by Berrisford *et al.*) These changes in the lower surface pressure are of course related to vortex stretching effects of the total fluid column above a given point on the surface (see also section 5.)

Our boundary condition on $(\partial\psi/\partial\theta)$ is dependent on time, because ψ generally will change in time, and dependent on horizontal position, because η is generally a nonconstant function of the horizontal coordinates. Formally, this no problem, but if we want to solve the inversion with the standard technique of separating the vertical coordinate this leads to time- and position-dependent vertical modes. We can render the lower boundary condition independent of position and time with a procedure analogous to the well-known construction of "surface charges" introduced into quasigeostrophic theory by Bretherton (1966). We will explicitly present this technique here, so it becomes clear that there is no problem in including the non-Doppler term in the lower boundary condition.

Define thereto a streamfunction $\tilde{\psi}$, that is related to the streamfunction ψ, by

$$\frac{\partial\psi}{\partial\theta} = \frac{\partial\tilde{\psi}}{\partial\theta} + \mathrm{H}((\theta_s + \epsilon) - \theta)\left(\frac{\psi}{\theta_s} - \frac{g\eta}{f_0\theta_s}\right), \tag{14}$$

where ϵ is an infinitesimally small positive temperature, which means that for all $\theta > \theta_s$ we can choose ϵ such that $\theta > \theta_s + \epsilon$. The function H is the Heaviside function, which is defined as

$$\mathrm{H}(x) = \begin{cases} 1 \quad , & x \geq 0 \\ 0 \quad , & x < 0. \end{cases} \tag{15}$$

The streamfunction $\tilde{\psi}$ is now defined up to an integration constant, which we will choose to be zero. As a result, the vertical derivatives of $\tilde{\psi}$ and ψ differ by a discontinuous function. On the other hand, the $\tilde{\psi}$ and ψ themselves differ by a continuous function, that has a nonzero value only below $\theta_s + \epsilon$. Because we choose ϵ infinitesimal, this means that this difference vanishes, even at θ_s. So we have

$$\begin{cases} \tilde{\psi} = \psi \quad , & \theta_s \leq \theta \leq \theta_t \\ \frac{\partial\tilde{\psi}}{\partial\theta} = \frac{\partial\psi}{\partial\theta} \quad , & \theta_s < \theta \leq \theta_t. \end{cases} \tag{16}$$

The streamfunction $\tilde{\psi}$ is chosen such that at the lower boundary we can satisfy Eq. 13 if we take

$$\left(\frac{\partial\tilde{\psi}}{\partial\theta}\right)_s = 0, \tag{17}$$

which is, contrary to the boundary condition on $(\partial\psi/\partial\theta)$, trivially independent of position and time.

The quasigeostrophic potential vorticity can also be expressed in terms of $\tilde{\psi}$. The vertical derivative of the rightmost term in Eq. 14 contributes a Dirac-delta function to the quasigeostrophic potential vorticity:

$$q = f + \mathcal{L}\tilde{\psi} + \frac{f_0^2 \rho_0}{(dp_0/d\theta)} \left(\tilde{\psi} - \frac{g\eta}{f_0}\right) \delta(\theta - (\theta_s + \epsilon)). \tag{18}$$

We have omitted contributions to q that are proportional to $\mathrm{H}((\theta_s+\epsilon)-\theta)$ because formally the inversion of $\mathcal{L}$ can be written as an integral of q multiplied by a Green function. The contributions that have been omitted vanish for infinitesimal ϵ.

Equations 4 and 18 now form a closed system under the homogeneous vertical boundary conditions 10 and 17. It should be emphasized that this system is equivalent to the original quasigeostrophic system, where the potential vorticity was given by Eq. 5, and the lower boundary condition by Eq. 13. The price one pays for simplifying the lower boundary condition is the inclusion of a potential vorticity sheet just above the lower surface.

The isentropic coordinates allow for an interpretation of this surface sheet in terms of a "physical" realization. The sheet has an infinitesimal vertical extend in isentropic coordinates but the pressure makes a finite jump (as demanded by Eq. 8) over the sheet. This means that the sheet may be interpreted as a layer of air with very low static stability — so is very thin in isentropic coordinates — and which mass content is in accord with the pressure jump over the sheet. The mass content of the sheet is prescribed such as to compensate the mass variation of the atmosphere above the sheet, so that all pressure variations at the bottom of the sheet vanish, as expressed by Eq. 17. The surface sheet carries finite quasigeostrophic potential vorticity, because Eq. 5 contains a term proportional to $\partial p'/\partial\theta$, which, in turn, contributes a Dirac-delta function to the quasigeostrophic potential vorticity.

It may be tempting to relate the surface sheet to a planetary boundary layer, which also has very low static stability. However, there are two ways in which the planetary boundary layer is fundamentaly different from this surface sheet. Firstly, the mass content of the planetary boundary layer is mostly determined by the turbulent conditions in the boundary layer itself; it does not passively adjust to the mass content of the atmosphere above. Secondly, in our formulation the planetary boundary layer should be interpreted as a disturbance to a finite $dp_0/d\theta$. But $(\partial p'/\partial\theta)/(dp_0/d\theta)$ becomes very large in the planetary boundary layer, which is forbidden in quasigeostrophic scaling.

4 Vertical modes

The inversion of the elliptic operator $\mathcal{L}$ can be performed partially by separation of the vertical coordinate. Thereto we will write the dynamical fields ψ, q, and $\mathbf{v}$ as a sum over vertical modes, like for example

$$\psi(x, y, \theta, t) = \sum_{m=0}^{\infty} \psi_m(x, y, t)\chi_m(\theta). \tag{19}$$

The vertical modes χ_m are eigenfunctions of the Sturm-Liouville equation

$$f_0^2 \frac{d}{d\theta}\left(\rho_0\theta\frac{d\chi_m}{d\theta}\right) - \gamma_m \frac{dp_0}{d\theta}\chi_m = 0, \tag{20}$$

where the eigenvalues γ_m have a dimension of inverse squared length. Under the general boundary conditions $\alpha_i d\chi_m/d\theta + \beta_i\chi_m = 0$, where the subscript "$i$" stands for bottom surface (s) or top (t), the eigensystem has the following properties (e.g., Ledermann 1982):

- The eigenvalues γ_m can be ordered such that $\gamma_0 < \gamma_1 < \gamma_2 < \ldots$. They form an infinite set of real numbers that tend to infinity as $m \to \infty$.

- The eigenfunctions χ_m can be chosen such that they are orthonormal in the following sense

$$-\int_{\theta_s}^{\theta_t} \frac{dp_0}{d\theta}\chi_m\chi_n\, d\theta = (p_s - p_t)\delta_{n,m}, \tag{21}$$

  where $p_s = p_0(\theta_s)$, $p_t = p_0(\theta_t)$, and δ the Kronecker symbol. The normalization is such that the eigenfunctions are dimensionless.

- The series expansion, Eq. 19, is uniformly convergent and the coefficients ψ_m are expressible as

$$\psi_m = -\frac{1}{p_s - p_t}\int_{\theta_s}^{\theta_t} \frac{dp_0}{d\theta}\chi_m\psi\, d\theta. \tag{22}$$

The boundary conditions that we wish to impose in our specific case, can be deduced from Eqs. 10 and 17. They are

$$\left(\frac{d\chi_m}{d\theta}\right)_s = \left(\frac{d\chi_m}{d\theta}\right)_t = 0. \tag{23}$$

We can multiply Eq. 20 with χ_m and integrate over the vertical coordinate θ. Partial integration combined with the boundary conditions in Eq. 23 and the normalization in Eq. 21 then leads to the following expression for the eigenvalues γ_m:

$$\gamma_m = \frac{f_0^2}{p_s - p_t}\int_{\theta_s}^{\theta_t} \rho_0\theta\left(\frac{d\chi_m}{d\theta}\right)^2 d\theta. \tag{24}$$

This equation proves that the eigenvalues γ_m are nonnegative. In fact, the function that is constant as a function of θ is a solution of the Sturm-Liouville problem with the boundary conditions of Eq. 23. This eigenfunction has eigenvalue zero. Equation 24 then shows that this must be the eigenfunction with the lowest index. Using the normalization in Eq. 21 we can therefore deduce

$$\chi_0 = 1, \quad \gamma_0 = 0, \tag{25}$$

which represents the barotropic mode.

The evolution of the coefficients q_m can be obtained by projecting Eq. 4 on the m-th vertical mode.

$$\frac{\partial q_m}{\partial t} + \sum_{k,l=0}^{\infty} A_{klm}\mathbf{v}_{g,k}\cdot\nabla q_l = 0, \tag{26}$$

where the interaction coefficients A_{klm} are defined as

$$A_{klm} = -\frac{1}{p_s - p_t} \int_{\theta_s}^{\theta_t} \frac{dp_0}{d\theta} \chi_k \chi_l \chi_m \, d\theta. \tag{27}$$

Using the definition of q in Eq. 18 we can easily show that

$$q_m = f\delta_{m,0} + (\nabla^2 - \gamma_m)\psi_m - \frac{f_0^2 \rho_s}{p_s - p_t} \chi_m(\theta_s) \sum_{n=0}^{\infty} \chi_n(\theta_s)\psi_n + \frac{f_0 \rho_s g}{p_s - p_t} \chi_m(\theta_s)\eta, \tag{28}$$

where $\rho_s = \rho_0(\theta_s)$. Here we see how the decomposition of the fields into vertical modes has solved the vertical part of the inversion of the linear operator $\mathcal{L}$. The mth potential vorticity mode generally depends on the streamfunctions of all other modes, due to the inclusion of the surface sheet. In this sense the modes are not "pure", but can be made so by an appropriate linear combination of modes.

The vertically expanded equations are formally equivalent to the original equations because any vertical profile can be expressed as an infinite sum over vertical modes with coefficients as in Eqs. 19 and 22. To reduce the number of vertical degrees of freedom, one may choose to truncate the vertical representation of the field to a finite number of vertical modes. The truncated equations are obviously not equivalent to the original equations. The most serious difference is that the potential vorticity *with* surface sheet is written as a finite series of continuous functions. This means that the surface sheet contribution to the potential vorticity is only poorly represented in the truncated system.

5 The equivalent barotropic vorticity equation

We are now able to work out how the equations look like for a system that is truncated to the barotropic mode. Using Eqs. 26 and 28 the resulting equations can be written as (omitting the subscript "0" of the barotropic components)

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) q = 0, \tag{29}$$

$$q = f + \left(\nabla^2 - L_R^{-2}\right)\psi + f_0 \frac{\eta}{H}, \tag{30}$$

where we have introduced the Rossby radius of deformation L_R and the scale height H as

$$L_R = \left(\frac{p_s - p_t}{f_0^2 \rho_s}\right)^{1/2}, \quad H = \frac{p_s - p_t}{g\rho_s}. \tag{31}$$

Equation 29, with q defined as in Eq. 30, is known as the equivalent barotropic vorticity equation.

Our boundary conditions necessarily lead to a finite Rossby radius for the lowest index vertical mode. Implementing the boundary condition in Eq. 13 directly would lead to the conclusion that the model had no true barotropic mode; the lowest eigenvalue γ_0 would necessarily be nonzero. Implementing it by using a vorticity sheet at the bottom surface, while restoring homogeneous boundary conditions, as in Eqs. 17, in effect gives a true barotropic mode, as in Eq. 25, but in the dynamics of this barotropic mode a finite *effective* Rossby radius shows up. This leads to the conclusion that, under natural boundary conditions ($z' = \eta$), a quasigeostrophic model should never show true barotropic dynamics. This may give us a clue why barotropic models performed rather poorly in numerical weather prediction (Wolff 1958): the atmosphere does not show barotropic behaviour; the lowest order behaviour is necessarily equivalent barotropic.

The term $-L_R^{-2}\psi$ is known as the "Cressman term." In the early practice of numerical weather prediction it was originally introduced as a correction to the barotropic vorticity equation — which is obtained by setting $L_R^{-2} = 0$ — to suppress the excessive retrogression of the largest planetary waves in barotropic models (Cressman 1958). In the derivation as presented here the Cressman term naturally comes into the equations. It originates from the boundary condition Eq. 13 as a result of the allowed pressure variations at the lower boundary. Together with the orographic term in Eq. 30 it is related to vortex stretching effects. This can be understood if one rewrites Eq. 13, using Eqs. 8 and 31, as

$$p'(\theta_s) = f_0\rho_s\psi(\theta_s) - \rho_s g\eta = \frac{p_s - p_t}{f_0}\left(L_R^{-2}\psi(\theta_s) - f_0\frac{\eta}{H}\right). \tag{32}$$

The left-hand side of this equation is proportional to the variation of the total mass above a given point at the surface. So the stretching terms indeed express the variations in the potential vorticity due to the atmospheric mass content between θ_s and θ_t.

The relation between the Rossby radius of deformation and the scale height is given by

$$L_R = \frac{(gH)^{1/2}}{f_0}, \tag{33}$$

which is a familiar relationship that, for example, can also be found in quasigeostrophic theory of the shallow-water equations (Pedlosky 1979). We may estimate the Rossby radius and the scale height from Eq. 31. The parameters $p_s = 1000$ hPa, $p_t = 0$ hPa, $\rho_s = 1.2$ kg m^{-3}, $f_0 = 10^{-4}$s^{-1}, and $g = 10$ m s^{-2}, lead to $L_R \approx 2900$ km, and $H \approx 8.3$ km. This value of the Rossby radius is larger than the value used in the early numerical prediction models. These models used an effective Rossby radius of about 800 km. This Rossby radius was estimated by fitting the behaviour of the equivalent barotropic model to the behaviour of the long waves in the real atmosphere. A more recent survey of the Cressman term (Rinne and Järvinnen 1993) indicated, though, that a broad range of Rossby radii could give comparable skill to the models. But the most optimal choice seemed to be a radius that was a function of position.

One may wonder why the value of the Rossby radius, that is obtained from the quasi-geostrophic model, does not optimally describe the behaviour of the atmosphere. One of the reasons may lie in the quasigeostrophic approximation itself. The basic state is barotropic in nature (all background fields are functions of θ alone). One may therefore speculate that the behaviour of the quasigeostrophic model remains too barotropic in nature compared to the state of the atmosphere, which is highly baroclinic in nature, especially near the tropopause. Note, for example, that the higher baroclinic modes have a lower Rossby raius, which indicates that baroclinic behaviour may be associated with lower effective Rossby radii. Indeed, earlier derivations of the equivalent barotropic vorticity equation (e.g., Thompson 1961) used a single vertical mode with a baroclinic structure. The derived Rossby radii could be considerably smaller than the Rossby radius which was derived above. This single vertical mode was determined empirically.

The large Rossby radius of the lowest index mode has a somewhat limited relevance for the modeling of the troposphere, because boundary condition Eq. 10 is only valid for isentropic levels that are high in the stratosphere. This means that this Rossby radius corresponds to the vertically integrated behaviour of the complete atmosphere instead of the troposphere alone. We must introduce baroclinic modes, in order to be able separate the behaviour of the troposphere from that of the stratosphere. This is the subject of the next section.

6 Multi-mode systems

Truncating Eq. 26 to higher maximum mode indices gives rise to multi-mode systems, in which baroclinic behaviour is included by the introduction of further vertical degrees of freedom. Let us write out how the general two-mode system looks like. Here we include the barotropic mode and the first baroclinic mode.

The interaction coefficients in the two-mode system are

$$A_{000} = 1, \quad A_{100} = 0, \quad A_{110} = 1, \quad A_{111} = \alpha, \tag{34}$$

where α depends on the vertical structure. The other elements A_{klm} follow from these by the complete symmetry of the interaction coefficients for the permutation of the indices. The equations of motion are

$$\frac{\partial q_0}{\partial t} + v_0 \cdot \nabla q_0 + v_1 \cdot \nabla q_1 = 0, \tag{35}$$

$$\frac{\partial q_1}{\partial t} + v_0 \cdot \nabla q_1 + v_1 \cdot \nabla q_0 + \alpha v_1 \cdot \nabla q_1 = 0. \tag{36}$$

These two-mode equations are equivalent to certain two-layer equations. Define thereto

$$q_\beta = q_0 + \beta q_1. \tag{37}$$

We then have the two-layer equations

$$\frac{\partial q_{\beta_+}}{\partial t} + v_{\beta_+} \cdot \nabla q_{\beta_+} = 0, \tag{38}$$

$$\frac{\partial q_{\beta_-}}{\partial t} + v_{\beta_-} \cdot \nabla q_{\beta_-} = 0, \tag{39}$$

if

$$\beta_\pm = \frac{1}{2}(\alpha \pm \sqrt{\alpha^2 + 4}). \tag{40}$$

For these values of β the potential vorticity in each layer can be written as a function of the streamfunction $\psi_{\beta_\pm}$ in the two layers as

$$q_\beta = f + \nabla^2 \psi_\beta - \frac{(\chi - \beta_-)\Gamma_\beta + \gamma_1 \beta}{\beta_+ - \beta_-} \psi_{\beta_+} + \frac{(\chi - \beta_+)\Gamma_\beta + \gamma_1 \beta}{\beta_+ - \beta_-} \psi_{\beta_-} + \frac{g}{f_0} \Gamma_\beta \eta, \tag{41}$$

where we have introduced

$$\chi = \chi_1(\theta_s), \qquad \Gamma_\beta = \frac{f_0^2 \rho_s}{p_s - p_t}(1 + \chi\beta). \tag{42}$$

The terms with χ result from the inclusion of the non-Doppler term in the lower boundary condition.

For this two-layer system, we can write the streamfunction and potential vorticity fields as a sum of contributions of the two layers, instead of the two modes. This is done by expressing ψ as as sum of ψ_{β_+} and ψ_{β_-}, instead of ψ_0 and ψ_1. So we have

$$\psi = \psi_0\chi_0 + \psi_1\chi_1 = \psi_{\beta_+}\frac{\chi_1 - \beta_-\chi_0}{\beta_+ - \beta_-} - \psi_{\beta_-}\frac{\chi_1 - \beta_+\chi_0}{\beta_+ - \beta_-}. \tag{43}$$

We have numerically calculated the vertical structure of the modes and the layers for an atmosphere consisting of a troposphere and a stratosphere. The vertical stratification of the atmosphere was fixed by setting $\partial z_0/\partial\theta = 333$ mK^{-1} when 283 K $= \theta_s \leq \theta \leq$ 318 K and $\partial z_0/\partial\theta = 55$ mK^{-1} when 318 K $=\leq \theta \leq \theta_t =$ 483 K. The tropopause is thus represented by a discontinuous change of $\partial z_0/\partial\theta$ at a potential temperature of 318 K. The surface pressure was taken to be 1000 hPa, which results in a tropopause pressure of 189 hPa at 11.7 km height and a top pressure of 39 hPa at 20.7 km height. This stratification is typical for summer conditions at mid-latitudes.

The amplitudes of the normalized vertical modes with $m = 0, 1, 2, 3, 4$ are plotted in Fig. 1, both as a function of potential temperature and as a function of height. The corresponding eigenvalues are in Table 1. The eigenvalues increase more-or-less quadratically with index, which means that the inverse Rossby radii increase more-or-less linearly with index. This is the result of the index being a generalized wave number (the index equals the number of zero's of the corresponding mode) and the Rossby radius being proportional to a generalized vertical wavelength.

Index	Eigenvalue
0	0
1	6.04425
2	28.0487
3	49.8233
4	93.2223
5	151.439

Table 1: *The eigenvalues of the first six vertical modes in units of* $f_0^2\rho_s/(p_s - p_t)$.

The vertical structure of the layers in the two-layer model are given by Eq. 43. They are plotted in Fig. 2. The β_+ layer corresponds to the troposphere and the β_- layer corresponds to the stratosphere. The potential vorticity in the two layers equals

$$q_{\beta_+} = f + \nabla^2\psi_{\beta_+} - 1.7456 L_R^{-2}\psi_{\beta_+} + 0.5990 L_R^{-2}\psi_{\beta_-} + 1.1466\frac{\eta}{H}, \tag{44}$$

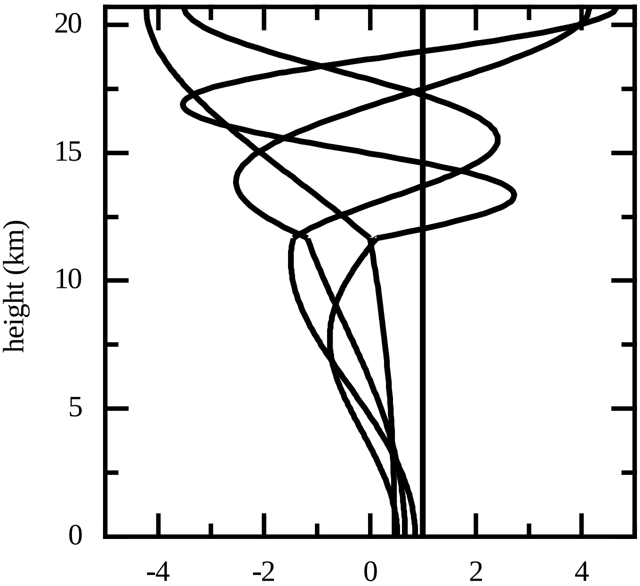

Figure 1: *Vertical profiles of the barotropic and first four baroclinic modes, plotted as a function of height.*

$$q_{\beta_-} = f + \nabla^2\psi_{\beta_-} + 5.0117L_R^{-2}\psi_{\beta_+} - 5.9760L_R^{-2}\psi_{\beta_-} - 0.4622\frac{\eta}{H}. \tag{45}$$

The effective Rossby radii are different for all streamfunction terms in the two layers. This is the result of the alternative derivation of the two-layer equations. If the same equations were to be derived from a vertical discretisation by finite differences of the quasigeostrophic potential vorticity, this asymmetry would vanish. The effective Rossby radii of the stratosphere are much smaller than those of the troposphere; in this model vortex stretching plays a much stronger role in the stratosphere than in the troposphere. The orography has the largest impact on the tropospheric layer as a vortex squeezing term, but it also has a direct but opposite effect on the stratospheric layer.

Truncations of Eq. 26 to a baroclinic mode with index M in general cannot be written as a $M+1$ layer system, as was the case for $M=1$ and, trivially, for $M=0$. This is related to the fact that the M-mode system for $M \geq 2$ has more degrees of freedom than the $M+1$-layer system. The degrees of freedom of the M-mode system correspond to the interaction coefficients A_{klm}, which for $M \geq 2$ have more independent values than the number of degrees of freedom for linear combinations of the q_m. For example, for $M=2$ we have 4 independent A_{klm} (namely A_{111}, A_{112}, A_{122}, and A_{222}), but only 2 independent coefficients for linear combinations of the q_m (namely $q_{\beta_1\beta_2} = q_0 + \beta_1 q_1 + \beta_2 q_2$). This asymmetry between mode-systems and layer-systems was already noted by Flierl (1978). He also showed that the excess number of degrees of freedom generally led to a better calibration of the mode system compared to the corresponding layer system, the best calibration corresponding to a direct solution of the Sturm-Liouville equation for a given background stratification.

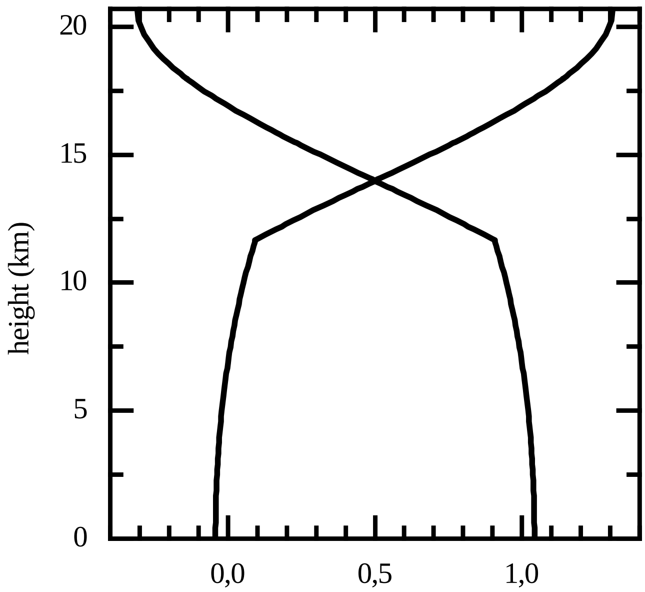

Figure 2: *Vertical profiles of the two layers, plotted as a function of height.*

7 Diabatic effects in the equivalent barotropic vorticity equation

In the quasigeostrophic approximation, the general expression for the influence of diabatic and frictional effects on the potential vorticity (e.g. Holton 1992) may be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) q = \frac{f_0}{dp_0/d\theta} \frac{\partial}{\partial \theta} \left(\dot{\theta} \left(\frac{dp_0}{d\theta}\right)\right) + \mathbf{k} \cdot \nabla \times \mathbf{F}_r. \tag{46}$$

Here $\mathbf{F}_r$ represents frictional effects, and $\dot{\theta}$ represents diabatic effects. This equation may also be projected on the barotropic mode to give the diabatic generalization of the equivalent barotropic vorticity equation. The result is

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) q = -\frac{f_0}{p_s - p_t} \left[\dot{\theta}_t \left(\frac{dp_0}{d\theta}\right)_t - \dot{\theta}_s \left(\frac{dp_0}{d\theta}\right)_s\right] + \mathbf{k} \cdot \nabla \times \mathbf{F}_{r0}, \tag{47}$$

with $\mathbf{F}_{r0}$ the projection of the frictional force onto the barotropic mode.

The diabatic effects will be chosen so weak, that the basic stratification remains unaltered. Under this constraint on the diabatic effects, the vertical modes do not change in structure because the Sturm-Liouville equation, Eq. 20, is only dependent on the basic stratification. Forcing at the lower and upper boundaries are also not of influence on the basic stratification. For example, we can introduce a forcing on the lower boundary without changing the value of θ_s, because θ_s is allowed to be somewhat different from the true lower boundary potential temperature Θ_s, as expressed by equation 12. The same considerations hold for the upper boundary.

As an example we will propose a simple forcing model, which represents a temperature relaxation toward some prescribed surface temperature distribution. The physical process that should be associated with this relaxation is the vertical turbulent exchange of latent and sensible heat between the planetary boundary layer and the rest of the atmosphere. The model for this relaxation will be written as

$$\dot{\theta}_s = -\frac{T'(\theta_s) - T'_R(\theta_s)}{\tau_R}, \tag{48}$$

where T'_R is the temperature variation towards which the temperature relaxes, and τ_R is a relaxation timescale. The right-hand side of this equation can be rewritten in terms of the streamfunction, by using Eqs. 2 and 3, as

$$-\frac{T'(\theta_s) - T'_R(\theta_s)}{\tau_R} = -\frac{f_0}{C_p\tau_R}(\psi(\theta_s) - \psi_R(\theta_s)) + \frac{g}{C_p\tau_R}(z'(\theta_s) - z'_R(\theta_s)). \tag{49}$$

The second term is an order of magnitude smaller than the first term on the right-hand side of this equation. This can be easily seen by linearizing the second term around Θ_s, as in Eq. 12. We then have

$$z'(\theta_s) - z'_R(\theta_s) = z'(\Theta_s) - z'_R(\Theta_s) - \theta'_s\left(\frac{\partial z'}{\partial\theta} - \frac{\partial z'_R}{\partial\theta}\right) \tag{50}$$

The first term on the right-hand side vanishes, because the level Θ_s was defined to be the earth's surface. The second term on the right-hand side is a second order term, because it is a product of two first order terms.

The lower boundary forcing in Eq. 47 can now be rewritten as

$$\begin{aligned}\frac{f_0}{p_s - p_t}\dot{\theta}_s\left(\frac{dp_0}{d\theta}\right)_s &= -\frac{f_0^2}{C_p\tau_R(p_s - p_t)}\left(\frac{dp_0}{d\theta}\right)_s(\psi(\theta_s) - \psi_R(\theta_s)) \\ &= \frac{1}{\tau_R}\frac{g^2}{C_p\theta_s N_s^2}L_R^{-2}(\psi(\theta_s) - \psi_R(\theta_s)),\end{aligned} \tag{51}$$

where in the last equality the definition of the Brunt-Väisälä frequency N is used. The term $g^2/C_p\theta_s N_s^2$ is a constant with a typical value of about 3. This constant can be absorbed into the definition of the timescale τ_R. Note that this relaxation term is proportional to the streamfunction deviation from a prescribed streamfunction. As the streamfunction follows from the potential vorticity via a smoothing operation, this means that the thermal relaxation contribution to the potential vorticity budget is a smooth contribution, which acts on the larger scales. A great advantage of this model is that it will hardly damp the smallest scales, so that the forcing of large-scale phenomena by small-scale eddies can be described using this model. Specific examples are the effect of Reynold's stresses on the large-scale flow, or the triggering of regime transitions by small-scale eddies. The equivalent barotropic vorticity equation with this temperature relaxation model has, for example, been used to illustrate the origin of the tropopause as the eroded edge of the polar vortex (Ambaum 1997). The same temperature relaxation model in combination with the barotropic vorticity equation, has proven to produce a realistic atmospheric model if judged on its average behaviour and its variability (Anderson 1995).

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