MULTIPLE STEADY STATES IN A CONTOUR DYNAMICS MODEL OF LARGE-SCALE ATMOSPHERIC FLOW

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1. INTRODUCTION

Two basic properties of geophysical flows are the material conservation of potential temperature θ and potential vorticity q, in the absence of frictional and diabatic processes. A natural view of the motions in the atmosphere and oceans should be based on θ and q, as eloquently argued by Hoskins et al. (1985). Two other properties are characteristic of the atmosphere and oceans: approximate hydrostatic equilibrium everywhere and approximate geostrophic equilibrium at high- and midlatitudes. Hoskins et al. (1985) emphasize that knowledge of the potential vorticity on surfaces of constant potential temperature (together with appropriate boundary conditions and the mass distribution under θ-surfaces) determines the flow completely, under the condition that the flow is in hydrostatic and geostrophic equilibrium (or in a more general form of 'balance'). This was realized by Charney (1948) who developed a systematic way of deriving the quasi-geostrophic system of equations. His derivation was explicitly based on the conservation of potential temperature and potential vorticity and the assumption of hydrostatic and geostrophic equilibrium.

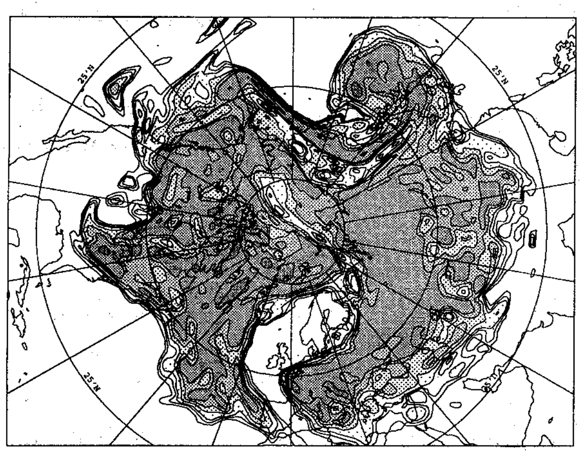

Fig. 1 Isolines of potential vorticity on the 320 K isentropic surface for February 14, 1994, 00 GMT. The contour interval is 1 PVU, or $10^{-6}m^2s^{-1}K\ kg^{-1}$.

Looking at the atmosphere in terms of potential temperature and potential vorticity gives a clear view which has only recently gained general acceptance. The contour dynamics approach to be discussed in this paper is explicitly based on this view. To demonstrate this we show in Fig. 1 the potential vorticity q on a surface of constant θ, where $\theta = 320$ K. We see that, over long zonal stretches, the potential vorticity is relatively uniform, except for a narrow band, where q changes rapidly. This observational fact leads naturally to the simplification that on a surface of constant θ the potential vorticity q is piecewise uniform. This will lead to a contour dynamics model as will be shown in the next section.

2. CONTOUR DYNAMICS

We will consider the evolution of potential vorticity on a single isentropic surface, say the 320 K surface in Fig. 1. It is assumed that this surface does not intersect the earth and that the surface can be approximated by a sphere with radius $a = 6.371 \times 10^6$ m, rotating with the earth's angular velocity $\Omega = 7.292 \times 10^{-5}$ s^{-1}. Distances will be expressed in units a, time in units Ω^{-1}, and points $\mathbf{r}$ on the sphere will be denoted by their geographical coordinates (λ, ϕ). The structure of q on this surface is assumed to be as sketched in Fig. 2.

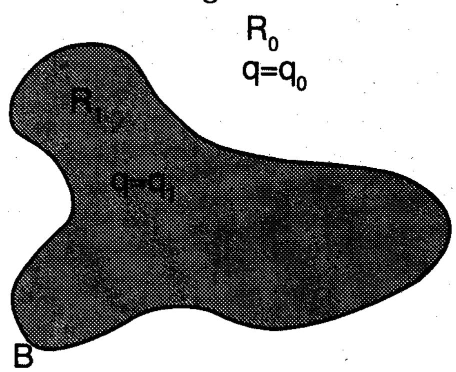

Fig. 2 Idealized distribution of potential vorticity q on a surface of constant potential temperature θ.

We assume that the time evolution of q is governed by the equivalent barotropic vorticity equation (F is the Froude number, f the Coriolis parameter, h the height of the orography and H a scale height):

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0, \tag{1}$$

$$\mathbf{v} = \mathbf{k} \times \nabla \psi, \quad q = \nabla^2 \psi - F\psi + f + \tau, \tag{2}$$

$$f = 2 \sin \phi, \quad \tau = f\eta, \quad \eta = \frac{h}{H}. \tag{3}$$

The streamfunction ψ (and thus the velocity $\mathbf{v}$) follows from q by inverting the Helmholtz operator $\nabla^2 - F$:

$$\psi = (\nabla^2 - F)^{-1}(q - f - \tau). \tag{4}$$

Writing the potential vorticity q as

$$q(\mathbf{r}) = q_0 + \begin{cases} q_1 - q_0 & \mathbf{r} \in R_1 \\ 0 & \mathbf{r} \in R_0 \end{cases} \tag{5}$$

it can be shown that

$$\nabla\psi(\mathbf{r}) = \nabla(\nabla^2 - F)^{-1}(q_0 - f - \tau) + (q_1 - q_0)\oint_B dl'[\mathbf{n}' \cdot \mathsf{T}(\mathbf{r};\mathbf{r}')]G(\mathbf{r};\mathbf{r}'), \qquad (6)$$

where dl' is a line element along the boundary B and $\mathbf{n}'$ is a unit vector locally perpendicular to the boundary and pointing away from R_1. In this expression T is a tensor and G is the Green's function of the Helmholtz operator for a sphere. We see that we have obtained a closed system in terms of the contour B. The system has two conserved quantities

$$A_1 = \int_{R_1} dS, \qquad (7)$$

$$E_t = \frac{1}{2}\int_S dS[\mathbf{v}^2 + F\psi^2]. \qquad (8)$$

These integrals can be transformed into contour integrals. The area A_1 and the total energy E_t will be used to catalogue stationary solutions. Details of this and the following sections can be found in Verkley(1994) and Ambaum and Verkley (1995).

3. STEADY STATES - INVISCID

We first observe that if the amplitude of the orography is zero, contours that coincide with a latitude circle are steady due to the zonal symmetry of the system. In a diagram in which these steady states are represented by their area A_1 and total energy E_t these states form a parabola-like curve, see Fig. 3a. A linear perturbation analysis shows that these states support linear Rossby waves. For some zonal flows one of these Rossby waves is stationary. At the corresponding points (resonances) in the area-energy diagram a set of steady states branches off that consists of stationary wave-like perturbations of finite amplitude with the same zonal wavenumber (m) as the linear stationary Rossby wave. These wave-like steady states form curves that lie somewhat below the curve of zonal states, see Fig. 3b. They can be obtained numerically by means of an iteration procedure.

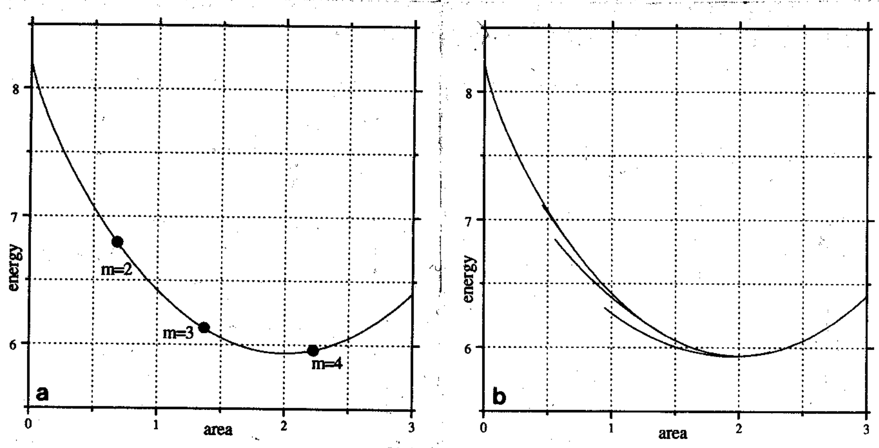

Fig. 3 (a) The area A_1 and the total energy E_t of zonal steady states of the system without orography. The dots mark resonances, that is, flows for which a linear Rossby wave (with zonal wavenumber as indicated) is stationary. (b) The same as (a) but with the branches of the wavelike steady states included.

The zonal flows mentioned above can be used as starting points in the search for steady states with orography. In the case of infinitesimal orography, we can linearize around the zonal contour and find the infinitesimal perturbation of the zonal contour that makes the flow stationary in the presence of orography. In accordance with earlier linear studies on the effect of orography, it turns out that every wave component of the orography induces a wave component at the contour of which the amplitude is inversely proportional to the phase velocity of the corresponding free Rossby wave. At the resonance points the linear analysis breaks down: here the resonant wave gets an infinite amplitude.

The linear treatment above maps every zonal flow into a slightly perturbed flow that is stationary in the presence of orography. This map can be generalized numerically to finite orography. Here we start with a particular zonal flow and an orography with small amplitude and find the corresponding steady state by means of a numerical iteration procedure. Once the steady state is obtained the amplitude of the orography is slightly increased and a new steady state is searched for using the previous result as a first guess. This is repeated until the orography has the required amplitude. This procedure generally works well; for nearly every zonal contour we obtain a nonzonal contour that is stationary in the presence of orography. The amplitude of the resulting solution depends on how close the original zonal contour is to a resonance: the closer to a resonance, the larger the amplitude. The position of the ridges and troughs relative to the orography depends on whether the original zonal contour is above or below the resonance. The different positions of these ridges and troughs also introduces a difference in energy. As a result, the diagram of steady states in the presence of finite amplitude orography - for which we used the T21 representation of the earth's orography - looks like Fig. 4a.

An enlarged part of Fig. 4a is shown in Fig. 4b. The curves in the upper left and lower right of the figure correspond to steady states that have no large meridional excursions and are therefore close to zonal. The branch in the middle of the figure, just below the zonal branch, is called the first blocking branch. The other branch in the center of the figure is called the second blocking branch. The latter two branches correspond to steady states with relatively large meridional excursions. Their difference is mainly a difference in phase relative to the orography. In the central part of the diagram we see that there are three different steady states for a given area. Examples are the states marked by dots in Fig. 4b. The corresponding flow patterns of these states are shown in Fig. 5.

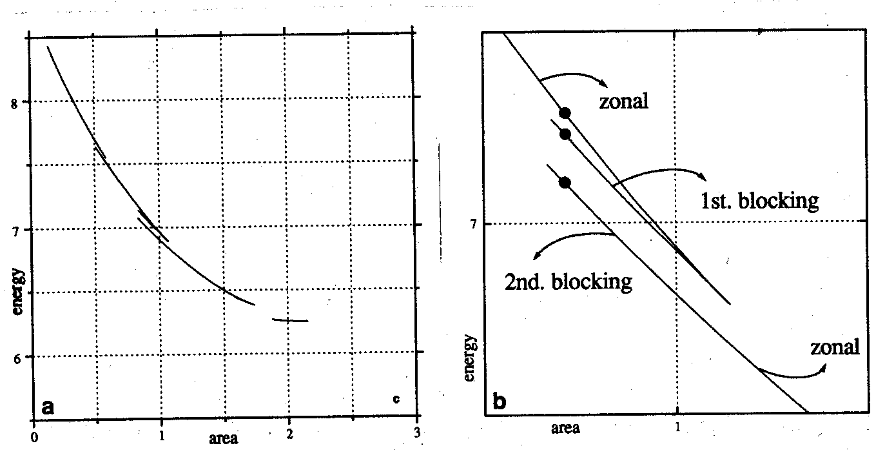

Fig. 4. (a) The diagram of steady states in the presence of the earth's orography in $T21$ truncation. (b) Enlarged part of Fig. 4a.

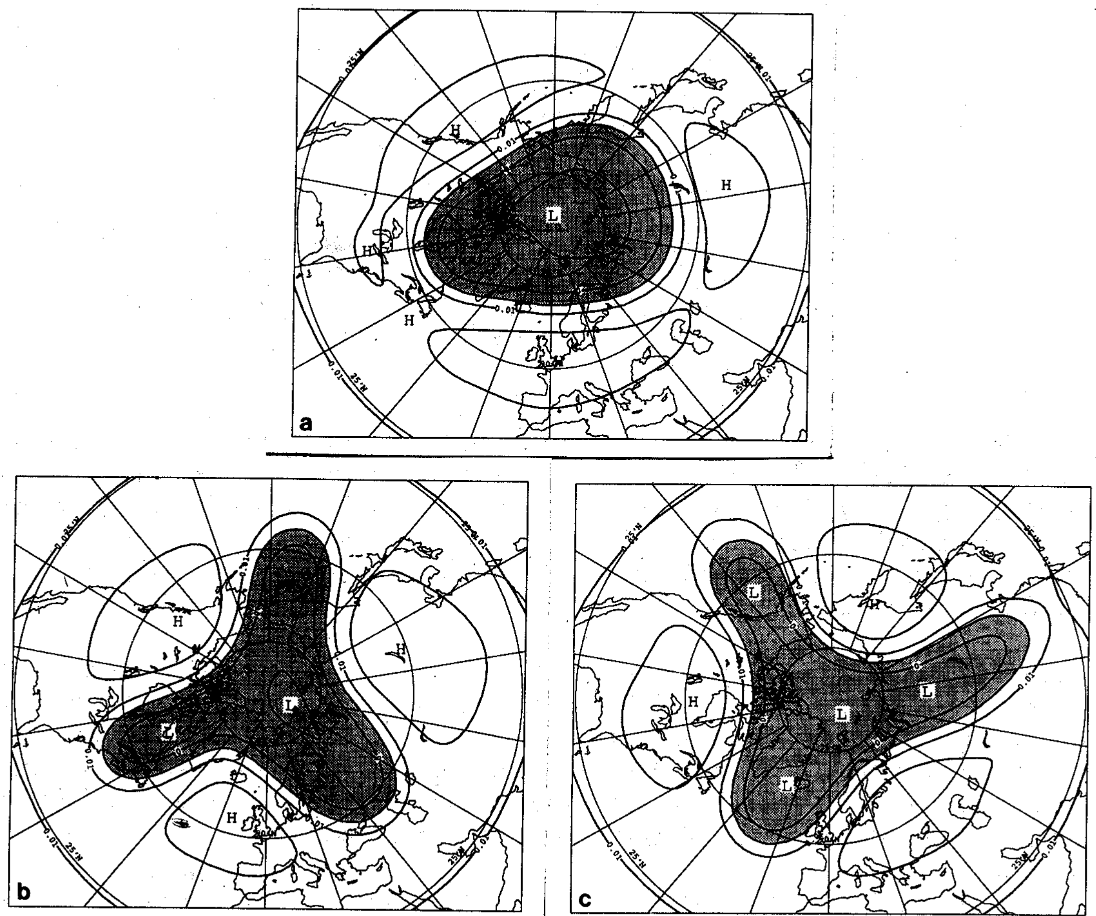

Fig. 5. The steady states corresponding to the three dots in Fig. 4b. The upper panel shows the flow on the zonal branch, the lower panel left the flow in the first blocking branch and the lower panel right the flow in the second blocking branch. The shaded region corresponds to the area of high potential vorticity. The solid lines are isolines of the streamfunction.

4. STEADY STATES - FORCING/FRICTION

The presence of forcing and friction, be it of mechanical or thermodynamic origin, means that on the right-hand side of (1) there is a source/sink term $\mathcal{S}$. Furthermore, a strict description of q in terms of a single discontinuity is no longer possible as it would not remain that way in the course of time. We replace the infinitely thin band of infinitely large gradients in q by a small, though finite, band (zone) in which the gradients of q are large but not infinitely large. Outside this band we suppose that q is uniform only in a statistical sense, assuming that all variations in q due to $\mathcal{S}$ are quickly homogenized by the shear associated with the large-scale flow. For the sake of argument, we will also assume that the structure of $\mathcal{S}$ is fixed in time and zonally symmetric. We then obtain the situation as sketched in Fig. 6.

Now, in the presence of the term $\mathcal{S}$ any isoline of q is advected by the velocity $\mathbf{v}_i$, given by

$$\mathbf{v}_i = \mathbf{v} + \mathbf{w}, \tag{9}$$

where $\mathbf{v}$ is the material fluid velocity and $\mathbf{w}$ is given by

$$\mathbf{w} = \mathbf{n}\frac{\mathcal{S}}{|\nabla q|}. \tag{10}$$

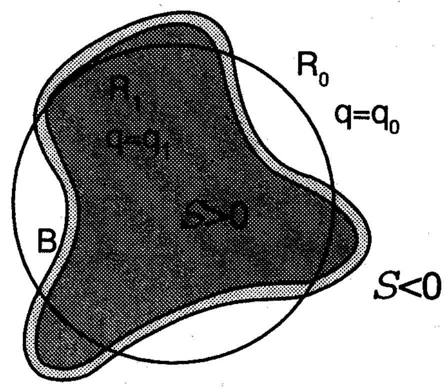

Fig. 6. Sketch of the structure of q in a system with a source/sink term $\mathcal{S}$. The potential vorticity q is assumed to be, in a statistical sense, uniform in the regions R_1 (darkly shaded) and R_0 (not shaded) separated by a small band (lightly shaded) in which q changes rapidly. For the sake of argument the source term $\mathcal{S}$ is assumed to be fixed in space and zonally symmetric, being positive to the north of a particular latitude and negative to the south of that latitude.

In the following we will apply (9) and (10) only at the contour in the center of the zone, assuming that the width of the zone does not change so that the gradient of q is fixed. If we now, in addition, assume that the average values of q in the two homogenized regions do not change (assuming that the integral of $\mathcal{S}$ over these regions vanishes) and that $\mathbf{v}$ is not affected by the finite width of the zone between R_0 and R_1, then (9) and (10) complete the contour dynamics system in the case of forcing and friction.

If we require that our central contour is stationary in the presence of $\mathcal{S}$, we should have on that contour

$$\mathbf{v}_i \cdot \mathbf{n} = 0 \Rightarrow \tag{11}$$

$$\oint_B dl \ \mathbf{v}_i \cdot \mathbf{n} = 0 \Rightarrow \tag{12}$$

$$\oint_B dl \ (\mathbf{v} + \mathbf{w}) \cdot \mathbf{n} = 0 \Rightarrow \tag{13}$$

$$\oint_B dl \, \mathbf{w} \cdot \mathbf{n} = 0 \Rightarrow \tag{14}$$

$$\oint_B dl \frac{\mathcal{S}}{|\nabla q|} = 0. \tag{15}$$

Note that (14) follows from (13) because the integral around the contour of $\mathbf{v} \cdot \mathbf{n}$, is the time rate of change of the area A_1 due to the velocity field $\mathbf{v}$. Because $\mathbf{v}$ is nondivergent, this rate of change vanishes. The condition (15) should hold for a finite as well as a infinitesimally small source/sink term $\mathcal{S}$. For a zonally symmetric distribution of $\mathcal{S}$ the condition in the latter case loosely fixes the *area* of a free stationary contour. This adds extra significance to the dots in Fig. 4b. Indeed, our provisional analysis indicates that forcing and friction reduce the space of steady states, in effect *isolating* them. The existence of isolated multiple steady states was discovered in the context of low-order spectral models of the atmosphere by Charney and DeVore (1979) and Wiin-Nielsen (1979). They proposed these states as models of atmospheric regimes. The fact that multiple steady states can be obtained, at least provisionally, in a more general contour dynamics model strengthens the case in favour of these states as models of atmospheric regimes.

5. SUMMARY AND CONCLUSIONS

Observed fields of potential vorticity q on surfaces of constant potential temperature θ naturally lead to an idealization in which q is assumed piecewise uniform. Combined with the assumption that the time evolution of potential vorticity is governed by the equivalent barotropic vorticity equation, the dynamics reduces to the dynamics of the discontinuity, that is, to contour dynamics.

In the case of no orography the stationary solutions of the system include zonal flows and families of finite amplitude m-fold symmetric waves. The latter families of solutions branch off from the set of zonal solutions at points where these zonal solutions are resonant, i.e. at points where the zonal solutions allow a stationary linear Rossby wave with zonal wavenumber m.

Stationary solutions in the presence of orography can be obtained numerically from initial zonal flows without orography by gradually increasing the amplitude of the orography and at the same time searching for a corresponding stationary flow. In this way stationary zonal flows without orography are mapped into stationary non-zonal flows with orography. In the neighbourhood of resonances this map becomes singular and would lead to wave-like solutions with ever increasing amplitude; there one generally obtains, for a given area, three stationary solutions that differ in total energy. One of these solutions is relatively zonal, the other two are characterized by large wave-like meridional excursions.

The presence of forcing and friction (mechanical or thermodynamic) leads to a source/sink term in the equation for potential vorticity. A provisional attempt to include this term in a contour dynamics model indicates that its presence reduces the space of free steady states, i.e. isolates them. If this can be substantiated, then it would corroborate the multiple steady states theory of atmospheric regimes pioneered by Charney and DeVore (1979) and Wiin-Nielsen (1979).

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