The three-dimensional vortical nature of atmospheric and oceanic turbulent flows

David G. Dritschel[a]
Mathematical Institute, University of St. Andrews, North Haugh, St. Andrews KY16 9SS, Scotland

Manuel de la Torre Juárez
Department of Earth and Planetary Sciences, Jet Propulsion Laboratory, 4800 Oak Grove Drive, Pasadena, California 91109-8099

Maarten H. P. Ambaum
Department of Meteorology, University of Reading, P.O. Box 243, Reading RG6 6BB, United Kingdom

(Received 20 March 1998; accepted 28 January 1999)

Using a novel numerical method at unprecedented resolution, we demonstrate that structures of small to intermediate scale in rotating, stratified flows are intrinsically three-dimensional. Such flows are characterized by vortices (spinning volumes of fluid), regions of large vorticity gradients, and filamentary structures at all scales. It is found that such structures have predominantly three-dimensional dynamics below a horizontal scale $L \approx \frac{1}{2} L_R$, where L_R is the so-called Rossby radius of deformation, equal to the characteristic vertical scale of the fluid H divided by the ratio of the rotational and buoyancy frequencies f/N. The breakdown of two-dimensional dynamics at these scales is attributed to the so-called ''tall-column instability'' [D. G. Dritschel and M. de la Torre Juárez, J. Fluid. Mech. **328**, 129 (1996)], which is active on columnar vortices that are tall after scaling by f/N, or, equivalently, that are narrow compared with L_R. Moreover, this instability eventually leads to a simple relationship between typical vertical and horizontal scales: for each vertical wave number (apart from the vertically averaged, barotropic component of the flow) the average horizontal wave number is equal to f/N times the vertical wave number. The practical implication is that three-dimensional modeling is essential to capture the behavior of rotating, stratified fluids. Two-dimensional models are not valid for scales below L_R. © *1999 American Institute of Physics.* [S1070-6631(99)02405-8]

I. SCIENTIFIC MOTIVATION

The motion of the atmosphere and oceans is tremendously complex, involving a huge range of spatial and temporal scales. Its prediction is one of the most challenging problems facing science today, and it is remarkable that broadly realistic, short-term forecasts can be made at all. A variety of physical, chemical, and dynamical processes compete to shape the motion, not all of which are well understood, and then there is the problem of observing and incorporating those observations into computer models.

In this article, we focus on the fundamental fluid-dynamical processes of atmospheric and oceanic flows. Physical and chemical processes like the supply of latent and sensible heat by the Earth's surface or the absorption of short wave radiation by ozone or long wave radiation by carbon dioxide are instrumental in shaping the observed *large scale* circulation of the atmosphere but tend to operate on much longer time scales compared to fluid-dynamical processes. The same is true for the oceans. These physical and chemical processes act to establish a flow which, however, is *dynamically* unstable. The flow continuously breaks down and reforms, the net result being the observed circulation.[1]

Fluid dynamical instability is thus seen to be at the heart of the problem. Rarely, though, are the atmosphere and oceans close to a basic state or equilibrium in any well-defined sense, certainly not at small to intermediate scales, which might best be described as turbulent. The turbulence there (below horizontal scales of 500 km in the atmosphere and 25 km in the ocean) is not pure in any sense, but is strongly affected by both rotation and density stratification (lighter fluid lying over denser fluid). Rotation favors the formation of ''deep'' flows having weak variations along the axis of the fluid's rotation (a result known as the Taylor-Proudman theorem), whereas stratification favors the formation of ''shallow'' flows having strong variations across stratification surfaces (isentropic, constant entropy surfaces in the atmosphere or isopycnal, constant density surfaces in the oceans) and motion parallel to these surfaces. These two effects have antagonistic consequences in the Earth's atmosphere and oceans, and, even to the present day, their combined role is not well understood.

Why does this matter? A cursory examination of the governing equations of motion suggests that these two effects are of the same order of magnitude around a horizontal scale $L = L_R$, where $L_R = NH/f$ is the so-called ''Rossby radius of deformation,'' H is a characteristic depth scale of the

[a]Electronic mail: dgd@mcs.st-and.ac.uk

fluid (about 7 km in the atmosphere and the full depth in the ocean), while N and f are the frequencies associated with stratification and rotation, respectively. N is the oscillation frequency that a small fluid volume would exhibit if it were displaced by a small vertical distance (for the ocean, $N = \sqrt{-g\rho^{-1}\partial\rho/\partial z}$ where ρ is the density and g is the acceleration due to gravity); this is called the buoyancy frequency and is here taken to be constant for purposes of argument though in reality it varies by more than a factor of 2 in both the atmosphere and the oceans. f is twice the local rotation rate of the fluid, i.e., the component of the rotation vector that projects on the local vertical (parallel to the effective gravity); this is called the Coriolis parameter and equals $2\Omega_e \sin\phi$ at the latitude ϕ, where Ω_e is the rotation rate of the Earth, and again f is here taken to be constant over scales L small compared with the radius of the Earth. For typical atmospheric and oceanic values of N and f, the Rossby radius $L_{\mathrm{R}} \approx 1000$ and 50 km, respectively, a scale which is much smaller than the Earth's radius.

The practical problem is that in numerical modeling scales smaller than L_{R} are only marginally resolved and thus subject to significant numerical dissipation [and note, according to Ref. 2, the breakdown of two-dimensional (2-D) vortical structures occurs at scales $L \lesssim L_{\mathrm{R}}/3$]. In recognition of the difficulty of properly resolving these scales, modelers parametrize the collective effects of unresolved scales by an enormously enhanced viscosity, called ''eddy viscosity,'' which, however, has little justification apart from ensuring numerical stability. Without knowing what actually occurs in this dynamically active, weakly dissipative range of scales, it is presently impossible to formulate a sensible parametrization for it. Yet, to adequately resolve these scales in present atmospheric and oceanic models is equally impossible: the present study suggests that a grid at least ten times finer is necessary, and this cannot be done without computers *at least* 10^3 times more powerful.

This article is concerned with the nature of rotating, stratified turbulence, a subject that has been debated now for nearly 30 years. The key points of this debate are reviewed in Sec III. This is proceeded by a short description of the physical system studied, the simplest one relevant to rotating, stratified turbulence. In Sec. IV we briefly describe the new numerical method[3] used in this study, and in Sec. V we present new—and to a great extent unexpected—simulation results obtained at exceptionally high numerical resolution. In Sec. VI we examine the time development of the energy spectra, in particular the characteristic vertical/horizontal scale ratio. Further implications of these results and their generalizations are discussed in Sec. VII.

II. THE PHYSICAL SYSTEM

We consider here only conservative motion, that is no forcing or dissipation. In this case, the fluid moves predominantly parallel to the basic stratification surfaces (which makes sense only if these surfaces do not overturn). This is an intrinsic characteristic of conservative rotating, stably stratified flows: layerwise two-dimensionality.[4,5] Under the *quasi-geostrophic* (QG) approximation,[6] these stratification surfaces are nearly flat (and never overturn), so the motion may sensibly be regarded as horizontal, i.e., with only two nonzero velocity components u and v in the x and y directions, respectively. The horizontal momentum equations, at leading order, simplify to a balance between the Coriolis force and the horizontal pressure gradient, this balance being called ''geostrophy'', while the vertical momentum equation reduces to hydrostatic balance. The momentum equations are used at higher order in two small, comparable parameters, the ''Rossby number'' R_0 and the ''Froude number'' F_r, along with mass conservation and incompressibility to derive a leading-order approximation to the *potential vorticity* (PV) q, a field involving the effects of both rotation and stratification which is conserved following fluid elements, i.e., $\partial q/\partial t + \mathbf{u}\cdot\nabla q \equiv Dq/Dt = 0$, under adiabatic conditions (for further details, see Ref. 4).

The Rossby number R_0 is defined as $|\nabla\times\mathbf{u}|/f$, where $\nabla\times\mathbf{u}$, the curl of the horizontal velocity $\mathbf{u}$, is the *vorticity* in the frame of reference rotating with the Earth, and $f = 2\Omega_e \sin\phi$. The Froude number F_r is defined as $|\mathbf{u}|/c$, where $c = NH$ is the characteristic speed of waves associated with the displacement of the stratification surfaces from equilibrium. The QG theory requires $\mathrm{F}_r^2 \ll \mathrm{R}_0 \ll 1$.

There are many situations when these parameters cannot be considered small, in equatorial regions (where $f\to 0$) or in the vicinity of strong topographic features, the upper atmosphere, or near strong thermal activity (where stratification surfaces may overturn). The QG equations cannot sensibly be used in weather prediction, as a result, but their inapplicability tends to be geographically localized; there are many situations of interest in which their use as a basic research tool is justified.[4,7–12] Likewise, their use here to study the fundamental properties of rotating, stably stratified turbulence is justified, since, according to comparative studies,[13] higher-order Rossby and Froude number effects are not expected to cause qualitative changes as long as these parameters are small compared to unity. [Notably, primitive-equation simulations of the tall-column instability performed by Dritschel (unpublished) show insignificant *quantitative* differences from QG simulations even for R_0 as large as 0.5.]

The expression for the potential vorticity q in QG theory is

$$q = f + \nabla_h^2\psi + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right), \tag{1}$$

where ψ is the streamfunction, from which the horizontal velocities can be determined using $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$. In Eq. (1), f is the Coriolis parameter [f_0 is the constant part of it, and $(f-f_0)/f_0 \lesssim O(R_0)$ for consistency], $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\rho_0(z)$ is the basic-state density profile with ''height'' z (in fact, z stands for log pressure, which is equivalent to geometric height for an isothermal atmosphere), and $N(z)$ is the buoyancy frequency profile. This set of equations is closed once boundary conditions are prescribed for ψ, and here we consider the simplest, most commonly employed conditions, namely $\partial\psi/\partial z = 0$ (uniform surface temperature) at the horizontal boundaries $z = 0$ and H.

These conditions enable one to invert the elliptic operator in Eq. (1) to obtain ψ from the instantaneous distribution of q, and then $\mathbf{u}=(u,v)$ by differentiation of ψ. One can then evolve q to the next instant of time by pure advection, $Dq/Dt=0$.

In this article, only the case of constant f, N, and ρ_0 is considered, following most previous works on QG turbulence. For the scales of present interest, which are small compared with the Earth's radius, it is highly accurate to consider constant f. Variable N deserves exploration, since it appears to favor more strongly baroclinic behavior, but this is beyond the scope of the present paper. Variable ρ_0, investigated in Ref. 1, also deserves further exploration, but its main effect is to favor vortices near the lower surface, where the fluid is relatively dense.

III. HISTORICAL SURVEY

Most previous studies of rotating, stratified turbulence have focused, almost exclusively, on the energy spectrum, the distribution of energy (kinetic+potential) with spatial scale. The spectrum reflects the relative importance of each scale within a turbulent flow, and its temporal change relates to the tendency for the flow to build up bigger structures (as in 2-D flows) or to fragment into smaller structures [as in three-dimensional (3-D) nonrotating, unstratified flows].

The association of these ''structures'' with vortices is commonly assumed; however, the spectrum itself does not contain the information necessary to make this association. Structures depend on phase correlations, information absent in the energy spectrum. The precise form of these phase correlations depends on the flow properties, i.e., whether or not the flow organizes into coherent, persistent spatial structures. Such organization appears to be an inherent characteristic of many flows, including rotating, stratified flows.[14] Furthermore, in such flows (including 2-D ones), the organized structures, that is, the vortices, appear to dominate in the sense that they contain most of the kinetic energy. What is left over is a largely passive sea of disorganized filamentary PV structure which is swept around the organized structures. This filamentary structure tends to cascade, at an exponential rate, to smaller and smaller scales where it is ultimately dissipated. It is replenished by sporadic interactions between the coherent, organized vortices (see Ref. 15 for further remarks in the 2-D case).

Little attention has been given to the physical characteristics of these coherent structures, and even less to the nature of their interactions, which is, in the end, responsible for the energy spectrum. Nonetheless, given the historical prominence of the energy spectrum, one can scarcely avoid a discussion of it.

Studies of atmospheric turbulence have primarily used the QG system of equations following Charney,[12] who found that they could be used to explain the observed form of the kinetic energy spectrum $E(\kappa)\sim\kappa^{-3}$ at mid-latitudes for longitudinal wavelengths $(2\pi/\kappa)$ between 1500 and 4000 km. He arrived at this result by assuming that

(a) one can ignore local variations of the Coriolis parameter f over the scales of interest,
(b) the vertical scale of the turbulence is small compared to the scale over which the buoyancy frequency N varies significantly,
(c) far from the boundaries, the turbulence is locally homogeneous and isotropic in horizontal planes, and
(d) nonlinear interactions are local in wave number space.

Assumptions (a) and (b) are well founded, but (c) and (d) are questionable for flows dominated, energetically, by coherent structures. Notwithstanding, using these assumptions, Charney proved that enstrophy (q^2) must in general cascade to smaller scales, whereas energy ''cascades'' to larger scales, where it is presumed to be dissipated (e.g., by thermal damping). Considerations along the lines used by Kraichnan[16] to infer the spectrum for 2-D turbulence led Charney to infer $E(k)\sim k^{-3}$ for QG turbulence, where k is the 3-D (total) wave number $k^2=k_x^2+k_y^2+k_z^2$, and here the z coordinate has been scaled by N/f. In this stretched z-coordinate system, the energy spectrum $E(k)$ is isotropic. Observations in the atmosphere at 500 hPa pressure (about 5 km in mid-latitudes) were shown to be consistent with this behavior.

As pointed out above, it is difficult to draw any conclusions about physical flow structures from this result. The physical interpretation of ''local, nonlinear wave number interactions'' and ''up-scale energy cascade'' has remained obscure, at best. The situation is made worse by the fact that the same inverse energy cascade and direct enstrophy cascade—as well as the k^{-3} spectrum—follow from a scale analysis of 2-D turbulent flows if k is taken as the 2-D wave number. This has been ascribed to the intrinsic quasi-two-dimensionality of atmospheric flows, and work has concentrated for decades on the strictly 2-D problem as a prototype of real atmospheric flows.[17]

A statistical interpretation of the 3-D isotropy found by Charney was put forward by Herring,[18] who developed a spectral transfer theory for an unbounded flow (infinite L_R) with energy injection at a prescribed wave number, k_0. Simulations of these equations indicated spectral isotropy at small scales, $k>k_0$, but strong anisotropy (characteristic of dominantly 2-D behavior) at large scales, $k<k_0$. However, in direct numerical simulations of the forced QG equations (in a triply periodic box), Métais *et al.*[19] found isotropy in the large-scale spectrum; moreover, they found that $E(k)\sim k^{-5/3}$ there.

In an effort to interpret Charney's result physically, laboratory experiments (and some associated numerical simulations) were performed by Colin de Verdière,[20] who confirmed the prediction that the dissipation of energy occurs principally at large scales (if a means of dissipation like thermal damping or surface friction is available there). The observed large-scale structures were 2-D and formed from an amalgamation of smaller-scale structures over time. The theory though says nothing about the physical form of any large-scale structures. Charney recognized that the presence of horizontal boundaries would violate the isotropy assumption at large scales, $L\gtrsim L_R=NH/f$, since at these scales, structures would be taller than the fluid depth H.

Still, an important question remained unresolved: given there are structures at small scales ($L<L_R$), what is their

form? Herring suggested that they are isotropic vortices. McWilliams[14] addressed this issue directly by developing a vortex identification procedure and applying it to a numerical simulation of QG turbulence (between rigid horizontal boundaries). From isotropic, very-small-scale initial conditions, he found the emergence of roughly ellipsoidal vortices having a height to width aspect ratio of about $1.6f/N$. This suggested that the coherent, small-scale spatial structure in QG turbulence is not isotropic, but nor is it completely anisotropic, i.e., 2-D. However, a more recent work by McWilliams and co-workers[21] modified this earlier conclusion. Using higher resolution and a triply periodic domain of equal dimensions (after scaling z by N/f, as usual), these authors illustrated that, at much later times than simulated previously, the ellipsoidal vortices align vertically into two tall, roughly columnar vortices having the same sign of PV throughout their depth, suggesting a final state consisting of (2-D) columnar vortices with small-scale 3-D disturbances superposed, in other words, a highly anisotropic final state.

This result appears to be inconsistent with the instability mentioned at the outset of this article. It is conceivable that the two-dimensionalization observed in Ref. 21 is due to the fact that these authors used periodic boundaries in the vertical, rather than rigid boundaries. However, such boundary conditions do not alter the fact that vortices of diameter $L \lesssim \frac{1}{3}L_R$ are unstable,[1] and indeed a recent study[22] has found that 2-D columns do break down even under periodic boundary conditions; moreover, the resultant small-scale vortices are close to isotropic. (Rigid isothermal boundaries, at which the vertical shear $\partial \mathbf{u}/\partial z \rightarrow 0$, enable vortices there to be taller than the f/N scaling would suggest.) A more plausible explanation for the behavior observed in Ref. 21 is that the initial conditions used and the accumulation of small-scale dissipation over a long period cause the energy to eventually fill the lowest wave numbers. By this time, the simulation ceases to be turbulent; a larger horizontal domain would allow the turbulence to be sustained for a longer period, perhaps indefinitely in the absence of dissipation.[22]

IV. THE NEW SIMULATION METHOD

Many numerical algorithms have been written to solve the QG set of equations. For the study of turbulence, the pseudo-spectral (PS) method, which employs the fast Fourier transform to take fields from physical to spectral space in order to deal with the nonlinear advection terms $(\mathbf{u}\cdot\nabla q)$, is without doubt the most popular algorithm. However, it has two serious limitations: (a) the time step used in the numerical integration of q is proportional to the grid size (to maintain numerical stability), and (b) enough dissipation must be introduced to prevent the build up of q near the grid scale. Insufficient dissipation leads to erroneous behavior or even numerical blow up. But, the dissipation employed does not just take away cascading PV as it passes the grid scale, it damps PV structures above grid scale as well. Moreover, because q is advected, its spectrum is shallow compared with other fields (like the velocity), and therefore dissipation at and above grid scale is particularly destructive.[23,24]

Recently, a means of overcoming these limitations was introduced by Dritschel and Ambaum,[3] who replaced the way q is evolved in the PS method by pure advection, i.e., solving $dx/dt=u$ and $dy/dt=v$ for points (x,y) lying on isolevels (contours) of q in each stratification surface (at each z). This is formally equivalent to solving $Dq/Dt=0$. In practice, a finite number of points are used to describe each contour, and a finite number of contours are used to describe the field q.[25] A finite number of stratification surfaces are used as well, just as in the PS method. The point is that pure advection can be carried out numerically without any constraint on stability, *and* representing q as contours permits one to resolve, at low computational cost, fine-scale structure such as sharp gradients and filaments at least an order of magnitude beyond the limits of the PS method. Eventually, fine-scale filamentary PV is removed by a topological reconnection scheme called ''surgery,''[25] but the result is a much reduced dissipation of PV and indeed a complete preservation of high-gradient (frontal) structures, structures that contribute most to the flow evolution.

This new method, called the ''contour-advective semi-lagrangian'' (CASL) algorithm, still makes use of the machinery in the PS method that gets the velocity field **u** from q (recall, one has to invert a 3-D linear operator, and a spectral method is here particularly efficient). But, in order to do this, q must be converted from its contour representation to a temporary grid representation (this is done by a fast domain-filling routine), and once **u** is obtained, it must be interpolated from the grid to the points on the contours. Using a dual representation for q in the same algorithm permits one to use a larger time step and to dissipate q in a far less drastic way than before. The result is a gain in computational efficiency of at least three orders of magnitude in simulations of QG turbulence[3] enabling us to perform simulations at unprecedented resolution.

V. SIMULATION RESULTS

Choosing an appropriate initial condition is the first task in any numerical simulation. Ideally, one would like to start with a flow which has been pushed, by slowly changing external influences, to the brink of instability, since it is difficult to imagine a flow getting far past this point, to a highly unstable state, without a qualitative change.[26] Turbulence may be triggered in this way, but one can also imagine a continually unsteady, 3-D state, one which is arguably more appropriate for the atmosphere and the oceans, flows which appear ever more unsteady with decreasing scale (cf. Ref. 27).

Here, we imagine turbulence arising from a down-scale cascade of PV [as is observed,[28] at least down to the 400 km scale ($\approx L_R/2$) in the atmosphere], the large scales being predominantly 2-D (vertically-coherent) in character. The 2-D flows generically exhibit this downscale cascade,[16] which is characterized in part by the production of an increasing number of small-scale vortices at ever finer scales[15,29] until dissipation becomes important. One may expect from Charney's[12] prediction some qualitative similarity.

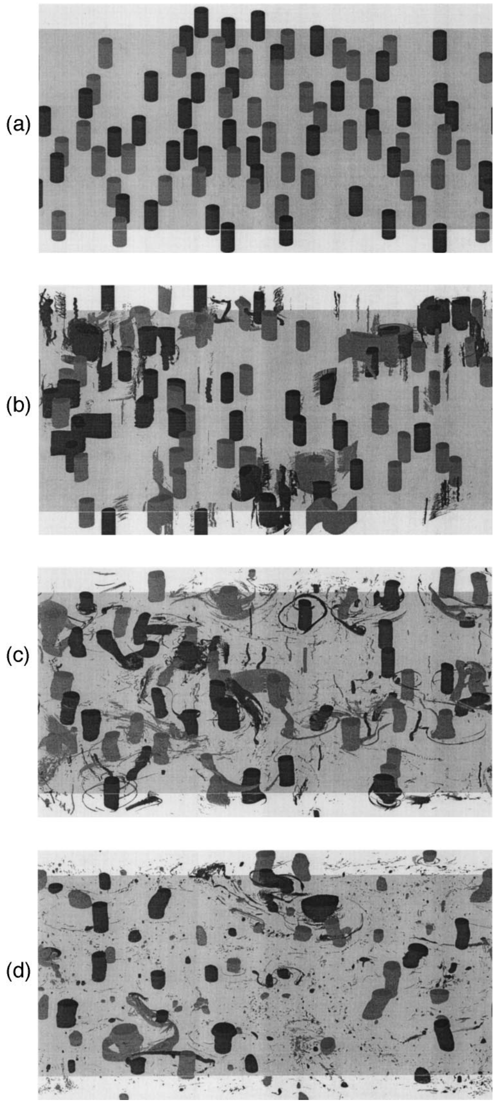

FIG. 1. Snapshots of simulation $\mathcal{B}$ at $t=0$, 14, 34, and 108 (time in units of $T=4\pi/Q$). The flow (the distribution of q') is viewed from an infinite vantage point in the plane $x=0$ at an angle of 60° from the zenith. Structures seen through the front face are faded, and the bottom of the domain is darkened. The first panel shows the initial condition, consisting of columnar vortices with 3-D perturbations that are too small to be visible in this picture. Note that initially 2-D interactions (second panel) produce tall columns that are unstable to 3-D disturbances. The third panel is near the maximum of complexity in the simulation (731 526 nodes, 26 755 contours). The fourth panel shows a typical ensuing state of the fluid, with a large amount of 3-D structure.

This motivates choosing a distribution of columnar vortices as the initial condition.

Vortices may be loosely defined as regions of anomalous $q'\equiv q-f$. This definition is made sharper by the fact that, when dissipative processes are very weak, vortices develop high gradients of q' at their periphery through the process of "vortex stripping."[30,2] High gradients are generic in these weakly dissipative flows,[23] to the extent that vortices, particularly small-scale vortices, are stripped of essentially all their peripheral vorticity. This motivates the use of vortex patches, regions of uniform q' bounded by infinite gradients.

Reference 3 has already presented a moderate-duration simulation of QG turbulence starting with such columnar tubes of uniform q'. We have performed additional, longer-duration simulations using virtually identical initial conditions, varying only the domain height, to quantify the flow characteristics. We have also performed simulations starting with vortices of varying sizes, as well as two starting with a 2-D jet (two adjacent walls of uniform, but opposite, q'). All of these simulations exhibit similar characteristics; the results are sensitive only to the ratio of the typical vortex scale L divided by the Rossby radius of deformation $L_R=NH/f$.

Three simulations are described, all starting with columns of equal diameter L and uniform PV anomaly $q'=\pm Q$ (with equal volumes of positive and negative q', filling a fraction $5\pi/256$ or 6.136% of the domain). The simulations differ principally in the ratio L/L_R; in simulation $\mathcal{A}$, $L=L_R/3$, the domain height $H=(f/N)3W/16$ (the domain width $W=2\pi$), and there are 20 columns initially; in simulation $\mathcal{B}$, $L=L_R/2$, $H=(f/N)W/16$, and there are 80 columns; in simulation $\mathcal{C}$, $L=L_R$, $H=(f/N)W/16$, and there are 20 columns. In simulation $\mathcal{A}$, the columns are marginally unstable, and one expects them to rapidly break down into 3-D volumes. However, in simulations $\mathcal{B}$ and $\mathcal{C}$, the columns are (initially) stable to 3-D disturbances. All simulations were conducted using a grid of $512\times512\times64$, with q' features kept to a scale ten times finer. The time step used was $\frac{1}{40}$. All simulations were taken to time $t=264$ (time is scaled on the vortex rotation period, $T=4\pi/Q$).

Figure 1 shows q' at a few selected times from simulation $\mathcal{B}$; lightly shaded structures have $q'=Q$, while darkly shaded ones have $q'=-Q$. The ratio of rotational and buoyancy frequencies f/N is used to scale the height coordinate z in all of the images displayed, so that the displayed height of the domain equals L_R. It can be seen that the initial 2-D flow breaks down into different types of vortical structures. The flow at late times is dominated by a set of semi-ellipsoidal vortices attached to the upper and lower surfaces and changes little in form after the time shown in Fig. 1(d). Only the widest vortices are able to resist the differential straining, and thus remain columnar over the whole depth of the fluid; they are usually very tilted, however. The rest of the fluid contains 3-D structures like vortex filaments (normally extending over more than one layer) and smaller ellipsoidal vortices. Similar observations were made also in Ref. 3, but the present simulations have been taken five times further to clearly reveal the structure of the emergent vortices and to demonstrate their long-time persistence.

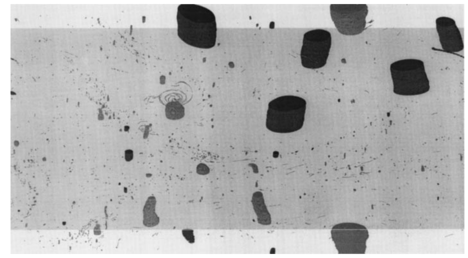

FIG. 2. Snapshot of simulation $\mathcal{C}$ at $t=192$.

Figure 2 shows a snapshot of q' in simulation $\mathcal{C}$ (L/L_R

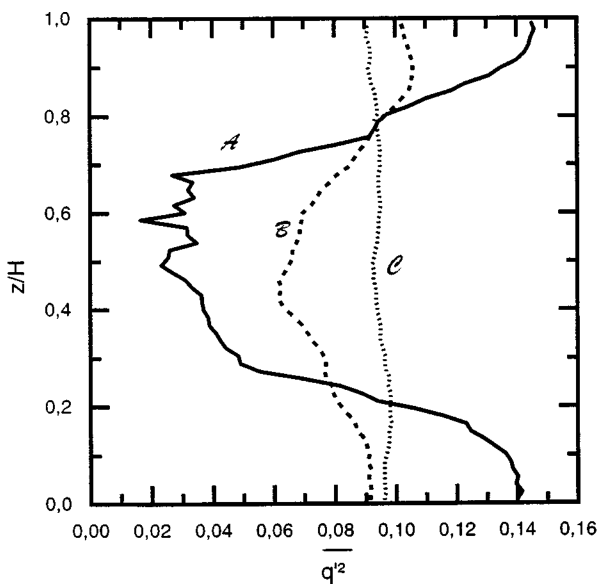

FIG. 3. The mean enstrophy $\overline{q'^2}$ as a function of scaled height z/H for simulations $\mathcal{A}$–$\mathcal{C}$ averaged over the period $250 \leqslant t \leqslant 264$. At $t=0$, $\overline{q'^2} = 5\pi/64 = 0.2454...$, independent of z. The inevitable dissipation of enstrophy is clear from this figure, though the dissipation is relatively low near the surfaces for simulation $\mathcal{A}$ in particular; the imposed boundary conditions (which imply $\partial \mathbf{u}/\partial z = 0$ at $z=0$ and H) on average allow for less shear near the surfaces.

$=1$ initially) at a late time $(t=192)$. The columnar vortices are initially stable, but due to 2-D vortex interactions, narrower (taller) columns are produced, which are again prone to instability. Again, a 3-D picture emerges, which is in essence not different from the other simulations: only the widest vortices remain columnar, but these show so much tilting, that also for these vortices one cannot speak of them as 2-D.

Figure 3 shows the enstrophy, the mean-square value of q' $(\int\int q'^2\, dx\, dy / \int\int dx\, dy)$, as a function of z/H, for all three simulations at late times. Initially, it is uniform, but, as a result of very fine-scale dissipation (at $\frac{1}{5120}$ of the domain width!) it decreases, especially in the middle of the domain for the $L=L_R/3$ case, so that by the end of that simulation, the enstrophy is primarily located near the upper and lower surfaces. With increasing initial vortex size, there is less dissipation of interior PV, and the dissipation becomes approximately uniform with height. The striking variation of enstrophy dissipation observed in the $L=L_R/3$ case is a consequence of the nonuniform vertical distribution of strain $\gamma = |\partial \mathbf{u}/\partial \mathbf{x}|$, which must be less, on average, at the surfaces on account of the isothermal boundary conditions there. (The survival of a vortex depends critically on γ/q' being small, less than about 0.1.[31–33] Greater ratios hasten the cascade of filamentary structures to small scales, where they are ultimately dissipated.) For wider initial vortices, ψ is more uniform with height, and the strain is dominated by horizontal derivatives of the velocity field.

The important point made by this set of simulations is that, even if initially all structures are sufficiently wide not to be prone to breakdown, interactions between the columns generate smaller columns and not just bigger ones, numerics permitting, as has been demonstrated repeatedly.[15,34,35] Thus, sufficiently narrow columns are inevitably produced, and, subsequently, they break down into 3-D vortical structures. Scales $L < \frac{1}{2} L_R$ are *intrinsically* three-dimensional.

VI. ENERGY SPECTRA AND SCALE RATIOS

The physical-space picture of rotating, stratified turbulence as a collection of large-scale, predominantly 2-D structures coexisting with small-scale, 3-D structures seems remote from the notion of isotropy first envisioned by Charney three decades ago. Of course, the simulations conducted here are strongly influenced by the top and bottom boundaries, so, strictly, one can only look for evidence of isotropy among the smaller-scale structures in the middle of the domain, to satisfy the conditions of Charney's theory. But then real atmospheric and oceanic flows have boundaries, so it is worth examining the nature of turbulence under these conditions.

Quite by chance, while trying to produce very simple diagnostics of the simulation results, we obtained a surprising result that is virtually independent of boundary influences. Our object was to examine the typical vertical to horizontal scale ratio λ in rotating, stratified turbulence, expecting mild anisotropy, with $\lambda \gtrsim 2f/N$, as suggested by physical-space images from these and many other simulations. We decided to quantify λ spectrally, in the following way. In a horizontally doubly periodic domain, here of dimensions $2\pi \times 2\pi$, the horizontal wave numbers k_x and k_y are integers. The vertical wave numbers, k_z, may be indexed by m starting from 0, and these generally depend on the depth of the domain H as well as vertical discretization adopted (see Ref. 2 for details). The k_z are also proportional to N/f. One may write $k_z = \alpha_m N/f$, $m=0,1,2,...$. For an infinite number of layers, $\alpha_m = m\pi f/(NH)$. The simulation results were Fourier analyzed to produce a horizontal energy spectrum $E_m(K)$ for each ''vertical mode'' m, where $K^2 \equiv k_x^2 + k_y^2$. The vertical modes, again for an infinite number of layers, are proportional to $\cos m\pi z/H$, so they are a Fourier series. Note that the $m=0$ mode is height independent—this is the ''barotropic'' mode. Since the vertical modes are orthogonal, the energy may be sensibly decomposed into parts due to each vertical mode, as we have done. In doing the Fourier analysis, $|k_x|$ and $|k_y|$ were truncated at $n_g/2$, where n_g is the basic grid resolution (512 here), and m at 64, the number of layers in each simulation. Several other different simulations were also analyzed and suggest that the results described below appear to apply to a broad class of flows.

Figure 4 shows the spectra E_m for the first ten vertical modes plotted against K at four selected times in simulation $\mathcal{B}$ (the one having $L/L_R = 1/2$). The first three times correspond to the last three images shown in Fig. 1, i.e., times $t=14$, 34, and 108, while the last is for $t=264$ (the end of the simulation). The barotropic energy E_0, shown in bold, dominates the other energy components at all times at large scales. Even at $t=264$, the total energy in the barotropic mode exceeds 90%. Even at the smaller scales, the barotropic component dominates. At early times $(t=14)$, the 3-D instability is just beginning and the ''baroclinic'' modes (m

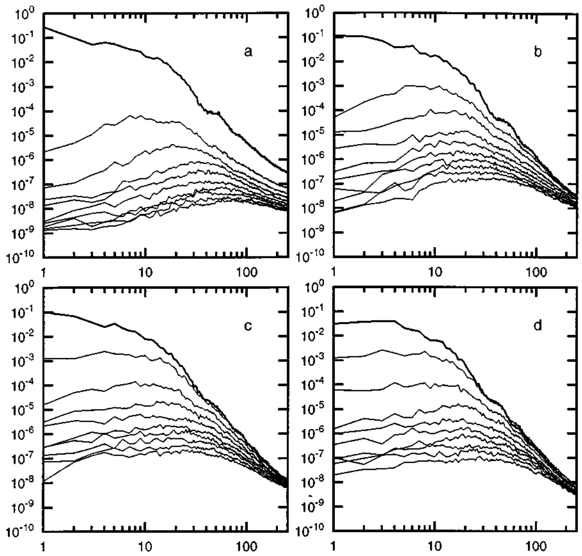

FIG. 4. The horizontal energy spectra $E_m(K)$ for the first ten vertical modes $m=0,1,\ldots,9$ in simulation $\mathcal{B}$ at (a) $t=14$, (b) $t=34$, (c) $t=108$, and (d) $t=264$. The barotropic spectrum $E_0(K)$ is shown in bold.

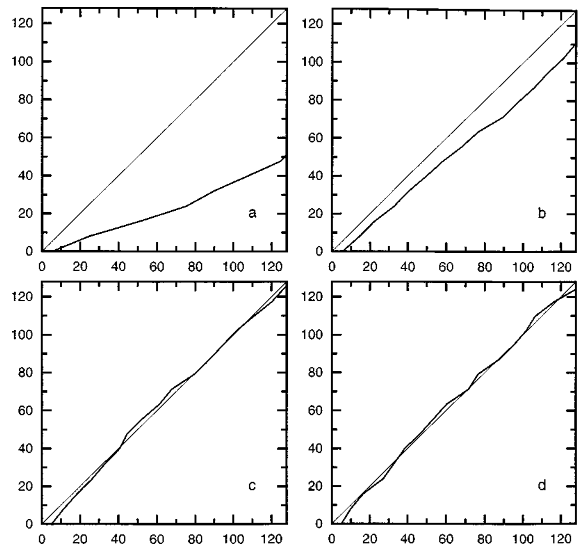

FIG. 5. The scaled vertical wave number $\alpha_m \equiv k_z f/N$ plotted against the mean horizontal wave number $K_m \equiv \Sigma_K K E_m(K)/\Sigma_K E_m(K)$ of each vertical mode m at the times shown in Fig. 4.

>0) are very weakly excited. In time, these modes grow and, particularly at small to intermediate scales, they become comparable to the barotropic mode. But there is more to be seen: at early times, the spectra are remarkably shallow, not badly approximated by a $K^{-5/3}$ behavior at intermediate K (though the large K spectra are influenced by the discrete nature of the data set—this generally leads to a shallowing for large K compared with the spectrum of a continuous function from which the discrete function may be thought to be a sample of). This behavior is consistent with a self-similar cascade from large to small scales, which here occurs by the initial production of smaller-scale structures during vortex interactions and 3-D breakdown. This initial cascade leads to a significant dissipation of the total energy (a 50% decrease by $t=100$), which is, however, not directly attributable to the 3-D dynamics—an analogous 2-D simulation (the same in every respect except for one layer only) exhibits a very similar energy decay. At later times, the spectra steepen to between K^{-4} and K^{-5}, indicative of a slow upscale ''cascade,'' during which the total energy is approximately conserved (it decreases by a further 10% between $t=100$ and $t=264$). Like in 2-D flows, the flow in this period is dominated by widely separated vortices which seldomly interact strongly. There is much less filamentary debris about than at early times, when vortex interactions were more frequent and energetic.

There is more still. Note that the peaks in the spectra shift to greater K with increasing m. We decided to quantify this behavior by computing $K_m \equiv \Sigma_K K E_m(K)/\Sigma_K E_m(K)$—this is the mean horizontal wave number for the vertical mode m. We plotted K_m versus α_m (the vertical wave number k_z multiplied by f/N), and the results, for the times shown in Fig. 4 are shown in Fig. 5. The surprising result is that $K_m \approx \alpha_m$ except at early times for all but the barotropic mode $m=0$. The relationship breaks down at wave numbers higher than 128, which is half of the maximum horizontal wave number, indicating that the smallest scale structures (below two grid lengths) are shallow compared to isotropy. It is as yet unclear whether this breakdown is real or a numerical artifact, although more recent simulations using higher vertical resolution indicate the latter. However, the fact remains that a high degree of correlation exists between K_m and α_m. This correlation may be taken as evidence that QG flows are isotropic, in the sense that vertical to horizontal scale ratios are equal to f/N. The barotropic mode behavior, however, stands out from this scaling relation. In physical space, one would expect it to enhance the spatial *anisotropy* of PV structures by extending them vertically, and this is consistent with the observed results.

VII. CONCLUSIONS

The results presented here underscore the inadequacy of 2-D models for simulating rotating, stratified flows. Previous work by Hua and Haidvogel[36] arrived at these conclusions for baroclinically unstable QG flows, i.e., flows forced by unstable surface temperature gradients, as a model of the Earth's troposphere. Here, we have found more generally, even for baroclinically stable flows, 2-D models are valid *only* in the absence of stratification and under rapidly rotating conditions. The presence of stratification fundamentally alters the behavior of the fluid at horizontal scales comparable to and below the Rossby radius of deformation, NH/f. Adequate vertical resolution is needed not only in equatorial regions, where $f \to 0$ and strong variations may occur across stratification surfaces, it is also needed to capture the inherent three-dimensionality of extra-tropical motions, which depends essentially on the ratio of rotational to buoyancy frequencies, f/N. Globally, it is reasonable to expect that the typical vertical/horizontal scale ratio of small to intermediate-sized vortices is comparable to the ratio of the rms absolute vorticity $\omega_a = |f\hat{\mathbf{e}}_z + \nabla \times \mathbf{u}|$ to N.

In present large-scale numerical models of the atmosphere and oceans, particularly in climate models, scales below the Rossby radius of deformation are only marginally resolved. Such scales are affected by numerical dissipation, which may be acting to suppress 3-D behavior by preventing the formation of smaller-scale vortices. To the extent that these scales, in the atmosphere, play a critical role in chemical mixing and therefore ozone depletion,[5,37,38] and, in the ocean, contain the greatest proportion of the flow's kinetic energy,[27] there is a clear need for vastly more efficient models, capable of resolving these scales. The CASL algorithm could be a prototype of such a model. One simply cannot afford to increase the resolution in present models to the required level; just to double the resolution requires at least eight times more computer power (and storage space), and already these models consume a significant proportion of the computer resources available on the fastest supercomputers. Much more than a doubling is necessary.

The fluid motion within the rotating, stratified atmospheres of other planets, in particular the gaseous outer planets, host a plethora of vortical structures, which are thought by some to be vertically shallow, 3-D, and by others to be deep, practically 2-D (see Ref. 39 for a review). The findings of the present work may help to resolve what form these structures take.

The extension of these results beyond quasi-geostrophy is a challenging problem, at present under scrutiny. There are indications[19,40] that turbulent flows in this regime tend to collapse back to quasi-geostrophy, perhaps because both anomalously tall *and* shallow structures tend to break down into preferentially isotropic structures. Though a careful study remains to be carried out, we conjecture that isotropy is a generic feature of rotating, stratified flows.

ACKNOWLEDGMENTS

The authors are grateful to Charlie Macaskill as well as to an anonymous referee for their care in reading this paper and for their insightful recommendations for revision. DGD is supported by the UK Natural Environment Research Council. Additional support was provided by the lsaac Newton Trust. MHPA received support from the EC funded project STREAM II. MTJ received support from the Spanish Ministerio de Educación *y* Ciencia, C.I.C.Y.T. project C95010502 and the NRC.

[1]M. H. P. Ambaum, ''Isentropic formation of the tropopause,'' J. Atmos. Sci. **54**, 555 (1997).
[2]D. G. Dritschel and M. de la Torre Juárez, ''The instability and breakdown of tall columnar vortices in a quasi-geostrophic fluid,'' J. Fluid Mech. **328**, 129 (1996).
[3]D. G. Dritschel and M. H. P. Ambaum, ''A contour-advective semi-Lagrangian algorithm for the simulation of fine-scale conservative fields,'' Q. J. R. Meteorol. Soc. **123**, 1097 (1997).
[4]B. J. Hoskins, M. E. McIntyre, and A. W. Robertson, ''On the use and significance of isentropic potential-vorticity maps,'' Q. J. R. Meteorol. Soc. **111**, 877 (1985).
[5]M. E. McIntyre, ''The stratospheric polar vortex and sub-vortex: fluid dynamics and midlatitude ozone loss,'' Philos. Trans. R. Soc. London **352**, 227 (1995).
[6]J. G. Charney, ''On the scale of atmospheric motion,'' Geofys. Publ. **17**, No. 2, 1–17 (1948).
[7]A. E. Gill, *Atmosphere-Ocean Dynamics* (Academic New York, 1982).
[8]M. N. Juckes and M. E. McIntyre, ''A high resolution, one-layer model of breaking planetary waves in the stratosphere,'' Nature (London) **328**, 590 (1987).
[9]J. R. Holton, *An Introduction to Dynamic Meteorology*, 3rd ed. (Academic, San Diego, 1982).
[10]D. G. Dritschel and R. Saravanan, ''Three-dimensional quasi-geostrophic contour dynamics, with an application to stratospheric vortex dynamics,'' Q. J. R. Meteorol. Soc. **120**, 1267 (1994).
[11]A. J. Simmons and B. J. Hoskins, ''Baroclinic instability on the sphere: Normal modes of the primitive and quasi-geostrophic equations,'' J. Atmos. Sci. **33**, 1454 (1976).
[12]J. G. Charney, ''Geostrophic turbulence,'' J. Atmos. Sci. **28**, 1087 (1971).
[13]G. K. Vallis, ''Potential vorticity inversion and balanced equations of motion for rotating and stratified flows,'' Q. J. R. Meteorol. Soc. **122**, 291 (1996).
[14]J. C. McWilliams, ''The vortices of geostrophic turbulence,'' J. Fluid Mech. **219**, 387 (1990).
[15]D. G. Dritschel, ''Vortex properties of two-dimensional turbulence,'' Phys. Fluids A **5**, 984 (1993).
[16]R. H. Kraichnan, ''Inertial ranges in two-dimensional turbulence,'' Phys. Fluids **10**, 1417 (1967).
[17]P. B. Rhines, ''Geostrophic turbulence,'' Annu. Rev. Fluid Mech. **11**, 401 (1979).
[18]J. R. Herring, ''Statistical theory of quasi-geostrophic turbulence,'' J. Atmos. Sci. **37**, 969 (1980).
[19]O. Métais, P. Bartello, E. Garnier, J. J. Riley, and M. Lesieur, ''Inverse cascade in stably stratified rotating turbulence,'' Dyn. Atmos. Oceans **23**, 193 (1996).
[20]A. Colin de Verdière, ''Quasi-geostrophic turbulence in a rotating homogeneous fluid,'' Geophys. Astrophys. Fluid Dyn. **15**, 213 (1980).
[21]J. C. McWilliams, J. B. Weiss, and I. Yavneh, ''Anisotropy and coherent vortex structures in planetary turbulence,'' Science **264**, 410 (1994).
[22]D. G. Dritschel and C. Macaskill, ''On the numerical simulation of quasi-geostrophic turbulence,'' Phys. Fluids (under review).
[23]H. B. Yao, D. G. Dritschel, and N. J. Zabusky, ''High-gradient phenomena in 2D vortex interactions,'' Phys. Fluids **7**, 539 (1995).
[24]D. G. Dritschel, L. M. Polvani, and A. R. Mohebalhojeh, ''The contour-advective semi-Lagrangian algorithm for the shallow-water equations,'' Mon. Weather Rev. **127**(5), (1999).
[25]D. G. Dritschel, ''Contour dynamics and contour surgery: numerical algorithms for extended, high-resolution modelling of vortex dynamics in two-dimensional, inviscid, incompressible flows,'' Comput. Phys. Rep. **10**, 77 (1989).
[26]D. G. Dritschel, ''A general theory for two-dimensional vortex interactions,'' J. Fluid Mech. **293**, 269 (1995).
[27]C. Wunsch and D. Stammer, ''The global frequency-wavenumber spectrum of oceanic variability estimated from TOPEX/POSEIDON altimetric measurements,'' J. Geophys. Res. **100**(C12), 24895 (1996).
[28]K. S. Gage and G. D. Nastrom, ''On the spectrum of atmospheric velocity fluctuations seen by MST/ST radar and their interpretation,'' Radio Sci. **20**, 1339 (1985).
[29]P. Bartello and T. Warn, ''Self-similarity of decaying 2-dimensional turbulence,'' J. Fluid Mech. **326**, 357 (1996).
[30]B. Legras and D. G. Dritschel, ''Vortex stripping and the generation of high vorticity gradients in two-dimensional flows,'' Appl. Sci. Res. **51**, 445 (1993).
[31]S. Kida, ''Motion of an elliptical vortex in a uniform shear flow,'' J. Phys. Soc. Jpn. **50**, 3517 (1981).
[32]D. G. Dritschel, ''The stability of elliptical vortices in an external straining flow,'' J. Fluid Mech. **210**, 223 (1990).
[33]S. P. Meacham, K. K. Pankratov, A. F. Shchepetkin, and V. V. Zhmur, ''The interaction of ellipsoidal vortices with background shear flows in a stratified fluid,'' Dyn. Atmos. Oceans **21**, 167 (1994).
[34]D. G. Dritschel and D. W. Waugh, ''Quantification of the inelastic interaction of two asymmetric vortices in two-dimensional vortex dynamics,'' Phys. Fluids A **4**, 1737 (1992).
[35]D. G. Dritschel and N. J. Zabusky, ''On the nature of vortex interactions and models in unforced nearly-inviscid two-dimensional turbulence,'' Phys. Fluids **8**, 1252 (1996).
[36]B. L. Hua and D. B. Haidvogel, ''Numerical simulations of the vertical

structure of quasi-geostrophic turbulence,'' J. Atmos. Sci. **43**, 2923 (1986).
[37] J. R. Holton, P. H. Haynes, M. E. McIntyre, A. R. Douglass, R. B. Rood, and L. Pfister, ''Stratosphere-troposphere exchange,'' Rev. Geophys. **33**(4), 403 (1995).
[38] J. Thuburn and D. G. H. Tan, ''A parameterization of mixdown time for atmospheric chemicals,'' J. Geophys. Res.-Atmos. **102**, 13037–13049 (1997).
[39] J.-I. Yano, ''A critical review on the dynamics of Jovian atmospheres,'' Chaos **4**, 287 (1994).
[40] P. Bartello, ''Geostrophic adjustment and inverse cascades in rotating stratified turbulence,'' J. Atmos. Sci. **52**, 4410 (1995).