Moist Static Energy : Definition, Reference Constants, a Conservation Law, and Effects on Buoyancy

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by

Jun-Ichi Yano

CNRM, Météo-France and CNRS, 31057 Toulouse Cedex, France

and

Maarten H. P. Ambaum

Department of Meteorology, University of Reading, UK

Corresponding author address: Jun-Ichi Yano, CNRM, Météo-France, 42 av Coriolis, 31057 Toulouse Cedex, France. E-mail: jiy.gfder@gmail.com.

abstract

Atmospheric thermodynamic variables are commonly computed under approximations. Although exact formulas are available, they are rarely used. This paper addresses some potential issues arising when using approximate formulas by taking the moist static energy as an example. In this case it turns out that the temperature dependence of latent heat must be accounted for. We demonstrate, also by taking the moist static energy as an example, that the fixed zero-point energies do not affect the budget of a thermodynamic variable. The use of an exact formula increases the surface value of the moist static energy from the standard value by 15 K under a typical tropical sounding. However, the change of the parcel buoyancy by using the exact formula is less dramatic, but not negligible. Considering only a qualitative role of CAPE in convection parameterization, the use of an exact formula is likely not be critical for the practical purposes, but for quantitative purposes we find discrepancies as large as 150–300 J/kg.

1. Introduction

Thermodynamic variables are often presented in approximate forms. Studies exist presenting the definitions of these variables in exact forms (Hauf and Höller 1987, Ooyama 1990, Marquet 1993, 2011, 2015, Marquet and Geleyn 2013). However, these exact definitions are rarely used in data analysis in the literature. The motivation of this study is to examine whether such exact definitions of thermodynamic variables have important consequences, or whether approximate relations may be sufficient.

We take the moist static energy (*cf.,* Marquet 1993, 2015) as a specific example in this study. We address the following issues: firstly we relate the exact definition of the moist static energy to the standard approximation (Sec. 2). Secondly, as it turns out, under a rigorous derivation, its definition contains undefined reference constants, which can only be defined by careful laboratory measurements or quantum–mechanical first–principle calculations to the limit of zero absolute temperature. An obvious question to ask is the role of these constants in order to preserve a conservation law associated with the moist static energy (Sec. 3). Thirdly, perhaps surprisingly, a large discrepancy between the exact definition and the standard approximate definition is found, reaching about 15 K for typical tropical values. But how serious is this discrepancy? In order to answer this question, we evaluate the parcel–lifted buoyancy based on these two definitions by invoking the conservation of moist static energy under this process (Sec. 4). We also address the change of the values in CAPE (convective available potential energy) as a result.

2. Definition

Moist static energy is conserved under moist adiabatic processes and under hydrostatic balance, thus it is a useful quantity in order to understand moist convection in the context of the large–scale dynamics. Under the standard approximate formulation, its specific value (value per total air mass) is defined by

$$h = c_{pd}T + q_v L + gz, \tag{1}$$

where c_{pd} is the heat capacity of the dry air at constant pressure, T the temperature, q_v

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the specific vapour value, L latent heat of vapourization of liquid water, g acceleration of the gravity, z the altitude.

The physical meaning of this definition is relatively intuitive: the first term is the dry–air enthalpy (or "heat content"), the second term is a potential contribution to the first term due to latent heating, and the last term is the gravitational potential energy. It is also intuitively expected that the sum of these three terms would be conserved under adiabatic and hydrostatic transformations. Hydrostatic balance ensures that any change in gravitational potential energy is compensated in a change of enthalpy through a change in pressure.

In deriving an exact expression, we may divide the above expression into the two contributions: the "proper" enthalpy (the first two terms), $\tilde{h}$, and the gravitational potential energy. Thus,

$$h = \tilde{h} + gz, \tag{2}$$

where

$$\tilde{h} \simeq c_{pd}T + q_v L, \tag{3}$$

which corresponds to a standard definition of enthalpy found in textbooks on thermodynamics, which are mostly concerned with laboratory–scale processes, where changes in gravitational potential can be neglected. When changes in potential energy are important, the quantity h becomes the conserved variable. Ambaum (2010, Sec. 5.4) calls h the generalized enthalpy. Under standard approximations (namely the ideal gas law for any gaseous components and incompressibility assumptions for the solid and liquid components), the generalized enthalpy reduces to what is normally called the moist static energy. Here, we will adopt this standard terminology, and call $\tilde{h}$ the enthalpy, and h the moist static energy.

Taking the ideal gas approximation, the specific enthalpy is given by a sum of contri-

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butions from dry air, h_d, water vapour, h_v, and the liquid water, h_l. Thus,

$$\tilde{h} = (1 - q_t)h_d + q_v\, h_v + q_l\, h_l, \tag{4}$$

where q_l is the specific liquid water, and $q_t = q_v + q_l$. Note that a contribution of ice is not considered here just for keeping the mathematical expressions simple.

Each specific enthalpy value is defined by

$$h_d = h_{0d} + c_{pd}T, \qquad h_v = h_{0v} + c_{pv}T, \qquad h_l = h_{0l} + c_{pl}T, \tag{5}$$

in which the first term is a reference constant, and the second term is a relative value proportional to the specific heat at constant pressure for each component. This simple form for the gaseous components assumes they are ideal gases, an assumption which is very accurate for atmospheric temperature and pressure ranges. The simple form for the liquid component assumes constancy its the specific heat capacity as well as incompressibility. Again, these assumptions are accurate for atmospheric temperature and pressure ranges. The specific enthalpies for vapour and liquid water are related to the latent heat by

$$L = h_v - h_l. \tag{6}$$

More strictly speaking, each expression in Eq. (5) must be written in the form

$$h_j = h_{0j} + \int_0^T c_{pj}(T')dT'$$

with the subscript j suggesting a component with the heat capacity, c_{pj}, at constant pressure defined as a function of temperature. Under this strict expression, the reference value, h_{0j}, has a clear physical meaning as an enthalpy value extrapolated to zero absolute temperature, excluding phase transitions; see Kittel and Kroemer (1980, Ch. 6). Eq. (5) can be considered a local approximation for this rigorous expression around a temperature range of interest.

Substitution of Eqs. (5) and (6) into Eq. (4) lead to

$$\tilde{h} = \tilde{c}_p\, T + q_v\, L + (1 - q_t)h_{0d} + q_t\, h_{0l}, \tag{7}$$

where

$$\tilde{c}_p = (1 - q_t)c_{pd} + q_t\, c_{pl} \tag{8}$$

Eq. (7) may be considered a final expression sought, as presented in *e.g.,* Marquet (1993, 2015). However, the given formula is hard to interpret in an intuitive manner: we would expect that the heat capacity, c_{pd}, for the dry air in Eq. (3) would be replaced by that for the total air (including contained water), which is equal to

$$c_p = (1 - q_t)c_{pd} + q_v\, c_{pv} + q_l\, c_{pl}. \tag{9}$$

However, instead of c_p, we get $\tilde{c}_p$ defined by Eq. (8). It is also not obvious why the total water is weighted by the reference constant h_{0l}, for liquid–water enthalpy in the last term. It might have been more intuitive to expect, h_{0v}, especially when the air is unsaturated, thus $q_t = q_v$.

Of course this counterintuitive outcome is a simple result of following standard algebra. The counterintuitive outcome may be easily alleviated when we notice that substitution of Eq. (5) into Eq. (6) leads to an integral version of Kirchhoff's equation for ideal gases (Ambaum, 2010, Sec. 3.6):

$$L = L_0 - (c_{pl} - c_{pv})\, T, \tag{10}$$

and

$$L_0 = h_{0v} - h_{0l}, \tag{11}$$

which can be seen as the linear extrapolation of the latent heat of evaporation to zero absolute temperature. For water vapour extrapolated from typical atmospheric temperatures, we find

$$L_0 = 3.14 \times 10^6\,\mathrm{J\,kg^{-1}}.$$

Recall that the latent heat of evaporation for typical atmospheric temperatures is approximately

$$L \simeq 2.5 \times 10^6 \, \mathrm{J\,kg^{-1}}.$$

Substitution of Eq. (10) into Eq. (7) finally leads to:

$$\tilde{h} = c_p \, T + q_v \, L_0 + h_{0d} + q_t \, (h_{0l} - h_{0d}). \tag{12}$$

This expression (12) is much easier to interpret against the standard approximate expression (3): an exact expression (12) is obtained from an approximation (3) by replacing c_{pd} by c_p and L by L_0. Furthermore, a reference constant value must be added. Although the value of the reference constant may remain somewhat counterintuitive, it is now clear that its value is determined by splitting it out into the dry–air and the water dependences on the reference constants. Note also that L_0 is about 25% larger than the typical expected value of latent heat, L, of evaporation. This will have a substantial effect on the calculated values of moist static energy, as going to be addressed in Sec. 4.

3. Reference Constants and a Conservation Law

The final result (12) above is fairly close to the standard approximate expression (3) except for the two additional terms involving the reference constants. The first reference constant term, h_{0d}, is really just a constant, and it does not change by any processes. Thus, h_{0d} can take any arbitrary value, and we may simply set $h_{0d} = 0$ without loss of generality. On the other hand, the second reference constant term is proportional to the specific total water, q_t. Thus, the value of this term changes with q_t, and it looks like that the choice of the value for $h_{0l} - h_{0d}$ would affect the whole budget for the moist static energy when rain falls out of the parcel under consideration (e.g., when considering pseudo-adiabatic ascent). We now examine how the value of $h_{0l} - h_{0d}$ affects the budget more precisely.

For this purpose, we take the total derivative of the moist static energy defined by Eq. (12):

$$d\tilde{h} = c_p \, dT + T \, dc_p + L_0 \, dq_v + (h_{0l} - h_{0d}) \, dq_t \tag{13}$$

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The heat capacity at constant pressure is not constant any more, but

$$dc_p = (c_{pl} - c_{pd})\, dq_t - (c_{pl} - c_{pv})\, dq_v \tag{14}$$

By substituting Eq. (14) into Eq. (13), and after some rearrangements by recalling Eq. (10), we obtain

$$d\tilde{h} = c_p\, dT + L\, dq_v + (h_l - h_d)\, dq_t \tag{15}$$

Here, note that the value of $h_l - h_d$ depends on the reference constants $h_{0l} - h_{0d}$. This is different from the dependence on reference constants of $L = h_v - h_l$; the latter is a measurable property of a given substance which only depends on the change in reference constants across the phase transition, not on the absolute value of the constants themselves, whilst $h_{0l} - h_{0d}$ is *not directly measurable* in classical experiments and is not a property of a given substance. It is an artificial property in the specific enthalpy budget resulting only from changing the mass ratio between the liquid and dry components.

Inspection of Eq. (15) leads to two conclusions:

(i) When the total water is conserved, and $dq_t = 0$, the constant, $h_{0l} - h_{0d}$, does not affect the budget.

(ii) When the total water is not conserved, and $dq_t \neq 0$, the constant, $h_{0l} - h_{0d}$, must be specified. Especially, when the total water is lost by precipitation, the constant, $h_{0l} - h_{0d}$, defines the rate, $(h_l - h_d)\, dq_t$, that the enthalpy is lost locally by transport associated with precipitation. Thus, the enthalpy transport rate by precipitation depends on a choice of the value of $h_{0l} - h_{0d}$.

In order to close the enthalpy budget, the enthalpy transport rate must be specified. For this very purpose, the value of $h_{0l} - h_{0d}$ must also be specified. However, we should also keep in mind that $h_{0l} - h_{0d}$ is only a reference value. As already remarked, $h_{0l} - h_{0d}$ is *not directly measurable*, and its value, otherwise, plays no role in any place of our thermodynamic calculations. As an amount of total water, dq_t, is transported from one position to another, an amount of enthalpy, $(h_{0l} - h_{0d})dq_t$, is also indeed transported.

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However, the above argument means that its actual value plays no other part of the thermodynamics. This value is required merely for a consistent counting of the enthalpy budget. It transpires that a consistency of the budget is maintained regardless of the value assigned for $h_{0l} - h_{0d}$. Thus, $h_{0l} - h_{0d}$ can also be specified in arbitrary manner.

The reason for $h_{0l} - h_{0d}$ not playing any part of the thermodynamics may be understood in the following manner: the process of precipitation simply separates out the precipitable water from the rest of the parcel. The process of precipitation ($dq_t = dq_l < 0$) can be divided in two steps: firstly the separation within the parcel of the precipitating liquid and the rest of the mixture in the parcel and, secondly, the removal of the precipitating liquid from the parcel. In the first instance nothing happens to the total parcel, but the enthalpy in the parcel can be allocated separately to the precipitable water (an amount of $-h_l \, dq_t$ per unit total mass) and to the rest of the mixture. In the second step of the precipitation process, the liquid and its enthalpy is removed from the parcel mixture, but in this step the mass fraction of the dry air in the parcel mixture has increased by an amount $-dq_t$. Any first order effect of changing mass fraction of the vapour is captured in the second term of Eq. (15), but is in any case small because it is proportional to dq_t. The net change in specific enthalpy, therefore, must be $(h_l - h_d) \, dq_t$. This change is merely reflecting the changed mass fractions that the constituents of the parcel contribute to the specific enthalpy constant of the mixture. There is no thermodynamic transformation involved, just a recognition that different mixtures will have different specific enthalpy constants.

Under these considerations, we can simply set $h_{0l} - h_{0d} = 0$ in the following analysis, so that the simple change of composition, does not affect the reference constant of the mixture.

4. Data Analysis

Our next goal is to examine the extent that any data analysis results are modified by using the exact definition (12) for the enthalpy instead of the standard approximate

definition (3). For this purpose, we use two data sets. The first is a mean Caribbean sounding for July–October (for the hurricane season) assembled by Jordan (1958), and as given by his Table 5. This sounding provides values from the surface (1015 hPa) to 30 hPa level with a vertical resolution of 50 hPa from 1000 hPa to 200 hPa, 25 hPa above 200 hPa, 20 hPa above 100 hPa, and 10 hPa above 60 hPa.

The second data set is over the Intensive Flux Array (IFA) during the TOGA–COARE (Tropical Ocean Global Atmosphere Coupled Ocean Atmosphere Response Experiment) Intensive Observing Period (IOP — 1 November, 1992 through to 28 February, 1993). The data set is processed at the State University of Colorado and available from the web (http://tornado.atmos.colostate.edu/togadata/ifa_data.html: Ciesielski *et al.* 2003). The data consists of surface values, and values from 1000 hPa to 25 hPa level with a vertical resolution of 25 hPa. Sounding is given every 12 hours.

Fig. 1 shows the vertical profiles of the moist static energy computed by the two definitions (12) and (3) of the enthalpy for the Jordan mean sounding. The change of the moist static energy by adopting the exact expression (12) increases its value by 15 K at the surface compared to the approximate expression (3). However, we also note that the local absolute value of moist static energy does not play a role in the budget, for the same reason as the constant, $h_{0l} - h_{0d}$, can be kept arbitrary as discussed in the last section. Note that moist static energy also introduces an arbitrary constant offset for the potential energy, which is typically chosen to be zero so that geopotential height (ϕ/g_0) and geometric height above the geoid coincide near the surface.

Therefore, we turn to examining the change of the lifting–parcel buoyancy by adopting the exact definition for the moist static energy. Recall that the parcel buoyancy is defined by a difference of the virtual temperature between the parcel and the environment (*cf.*, Roff and Yano 2002). Here, in computing the lifting–parcel buoyancy, the moist static energy is used as a conserved variable along the parcel lifting. The parcel is lifted from the 950 hPa level, as an approximate height for the top of the well–mixed boundary layer.

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Below the saturation level, the specific vapour value, q_v, is also conserved, thus the parcel temperature is evaluated in a straightforward manner. Above the saturation level, we set the specific vapour to the saturated value, thus $q_v = q_v^*(T)$, and also assume that the total water, q_t, is conserved. As a result, the specific liquid water is $q_l = q_t - q_v^*(T)$. Under these constraints, the parcel temperature at a given level is calculated by a Newton–Raphson method assuming the conservation of the moist static energy. The resulting buoyancy based on this computation leads to the reversible buoyancy, as no water falls out from the parcel. The alternative definition, called "pseudo-adiabatic" buoyancy, is obtained by setting $q_l = 0$ in the final result.

The obtained parcel buoyancy under these two definitions for the buoyancy (reversible and pseudo–adiabatic) and the two definitions for the moist static energy are shown in Fig. 2 for the Jordan sounding. The pseudo–adiabatic buoyancy decreases by 1 K by adopting the exact definition (12), and the reversible buoyancy decreases by a lesser extent. At face value the change in parcel buoyancy is not as dramatic as the change of the absolute value of the moist static energy. However, the relative change can be very large, and for the reversible case we even find locations where the sign of the buoyancy changes between the two definitions.

The vertical integral of the positive parcel buoyancy under these two definitions leads to reversible and pseudo–adiabatic CAPE (convective available potential energy). These two CAPE time series are plotted for the first 30 days of the TOGA–COARE IOP over IFA in Fig. 3(a) and (b) along with the precipitation rate in (c). These time series, again, confirm the conclusion from Fig. 2: though the change of CAPE values by adopting the exact definition (12) is clearly not negligible, considering an overall qualitative usefulness of CAPE as a measure of convective instability (see *e.g.*, Yano *et al.* 2013 for further discussions), we conclude that the use of the exact definition (12) may not be critical in such cases.

However, if using CAPE in a quantitative fashion, the adoption of the exact defini-

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tion (12) seems imperative. Figure 4 shows scatter plots of two types of CAPE for the whole TOGA-COARE period calculated between the approximate expression (Eq. 3: horizontal axis) and the exact expression (Eq. 12: vertical axis). It is seen that the exact expression of the reversible CAPE consistently gives a value 150 J/kg larger than the approximated expression (a). For the pseudo–adiabatic CAPE, the approximation (3) consistently over-estimates the value by 40% compared to the exact value. The maximum difference may reach 300 J/kg for large CAPE values.

5. Discussions and Conclusions

Exact definitions for the atmospheric thermodynamic variables are known in the literature. However, in practical data analysis, these definitions are rarely used, but instead, standard approximate versions are invoked. Then what kind of consequences would such approximations have? This paper has examined this question by taking the moist static energy as an example.

At first sight, the exact definition (7) for the moist static energy appears not quite physically intuitive as the standard approximate definition (3). Here, we have shown that a re–writing of the definition (12) makes it more physically intuitive. An important corollary of this re–writing is the crucial importance to take into account the temperature dependence in the latent heat.

As emphasized by Marquet (2015), when the absolute thermodynamic values are concerned, the reference constant values must also be carefully specified. However, in practical applications, only the relative values across phase transitions (latent heats) are of importance. We have explicitly shown that the conservation law for the moist static energy can consistently be defined without defining the absolute values of the enthalpy constants.

More practical consequences of using the exact definition or the standard approximate definition are further examined by computing both the lifting–parcel buoyancy value as well as CAPE from tropical data sets based on both definitions. We show that the modifications are relatively minor when only qualitative properties of the parcel buoyancy or CAPE are

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considered, especially in context of convection parameterization (*cf.*, Yano *et al.* 2013). For example, the actual value of CAPE for a real parcel would likely be more strongly affected by an entrainment rate, when the change of buoyancy by entrainment is also taken into account. The latter parameter is vastly uncertain. Nevertheless, the two definitions give quantitatively substantially different values, with the approximate equations typically underestimating the reversible CAPE by about 150 J/kg.

Acknowledgements

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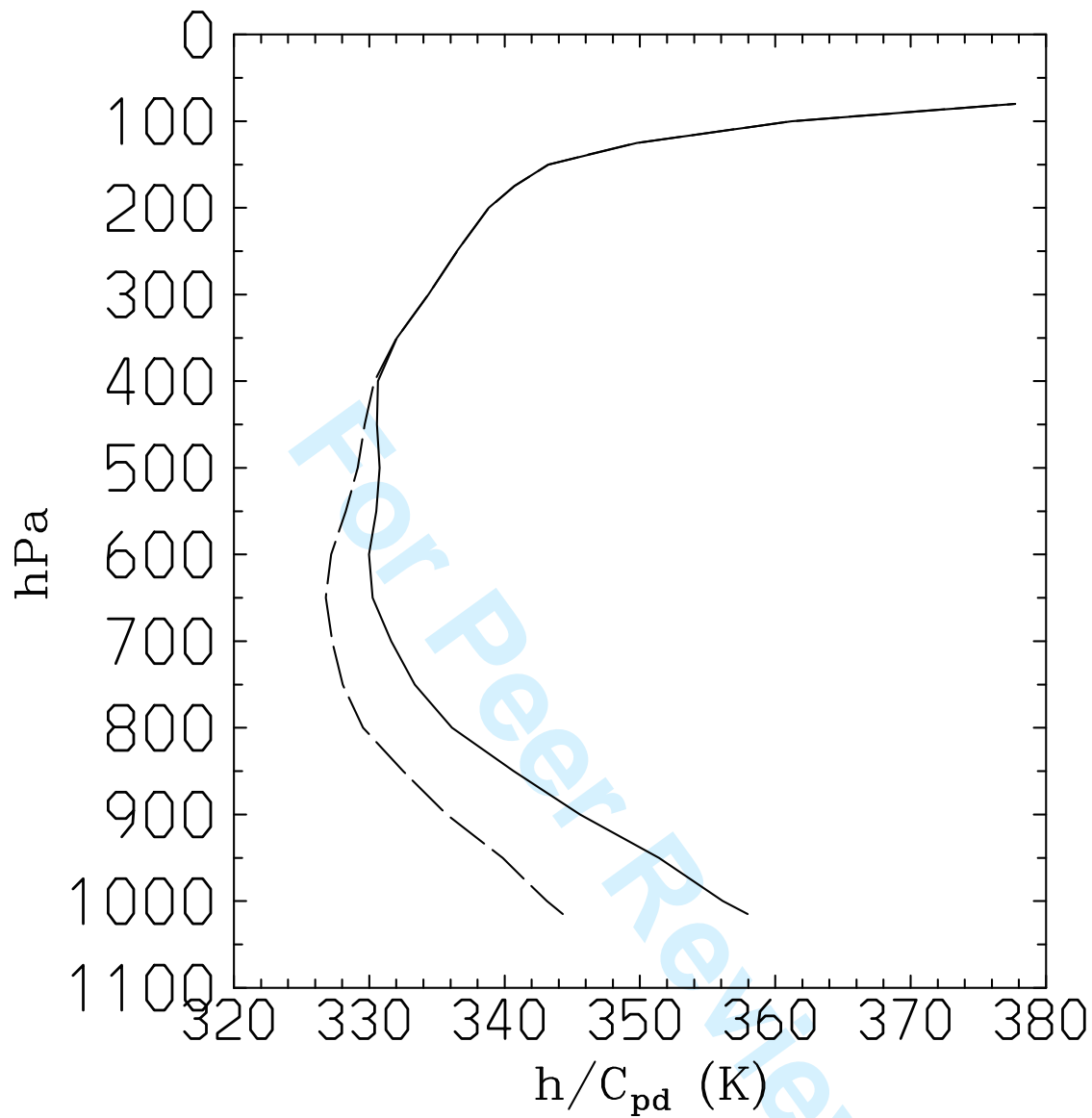
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Figures

Fig. 1: Vertical profiles of the moist static energy obtained under the Jordan sounding: the result based on the exact formula (Eq. 12: solid) and with the standard approximation (Eq. 3: long dash) for the enthalpy.
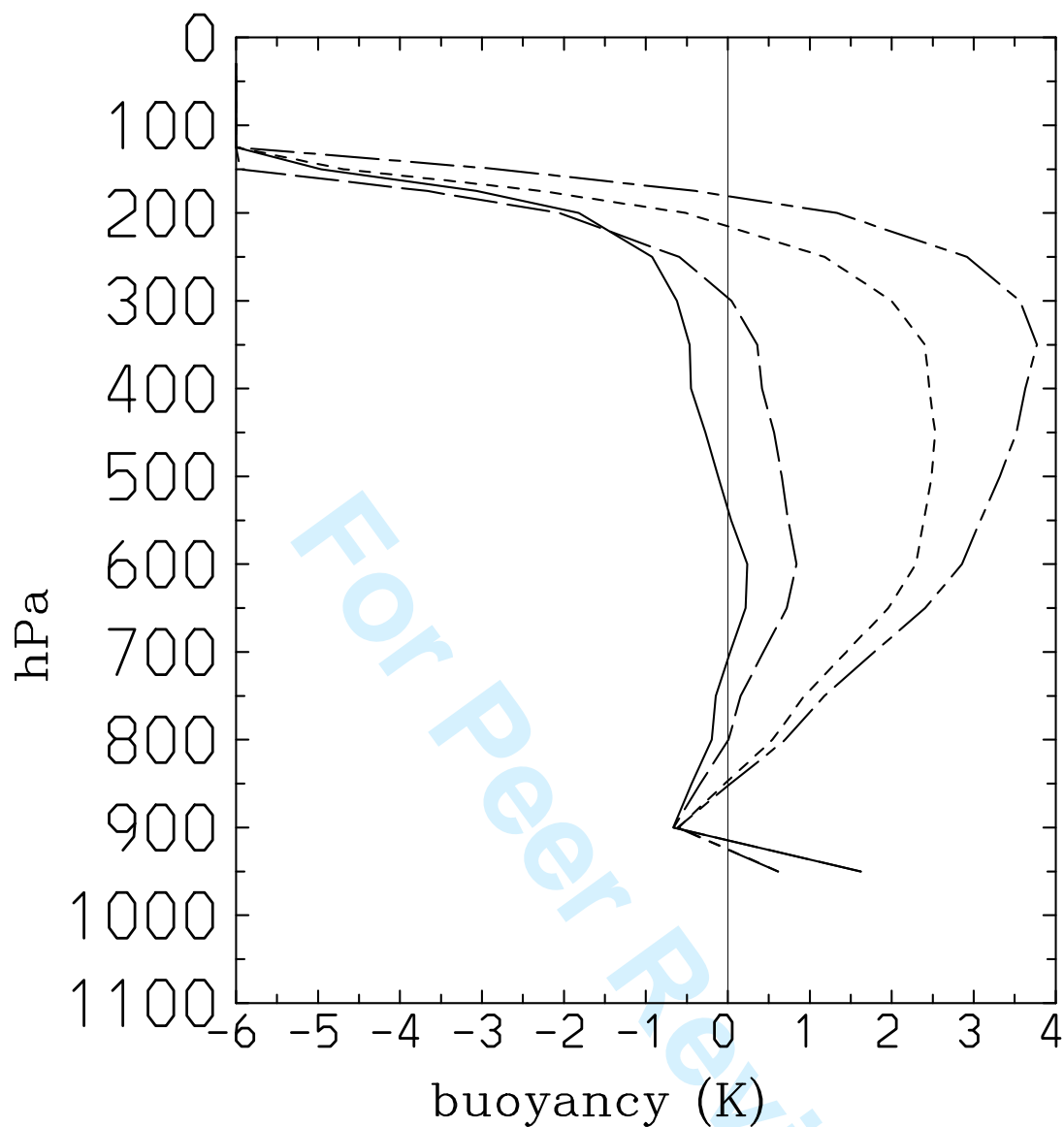
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Fig. 2: Vertical profiles of the parcel–lifted buoyancy under the Jordan sounding: the solid and long–dash curves (left side) show the reversible case, whereas the short–dash and chain–dash (right side) the pseudo–adiabatic case. The solid and short–dash curves are based on the exact formula (12), whereas the long–dash and chain–dash curves are under the standard approximation (3).
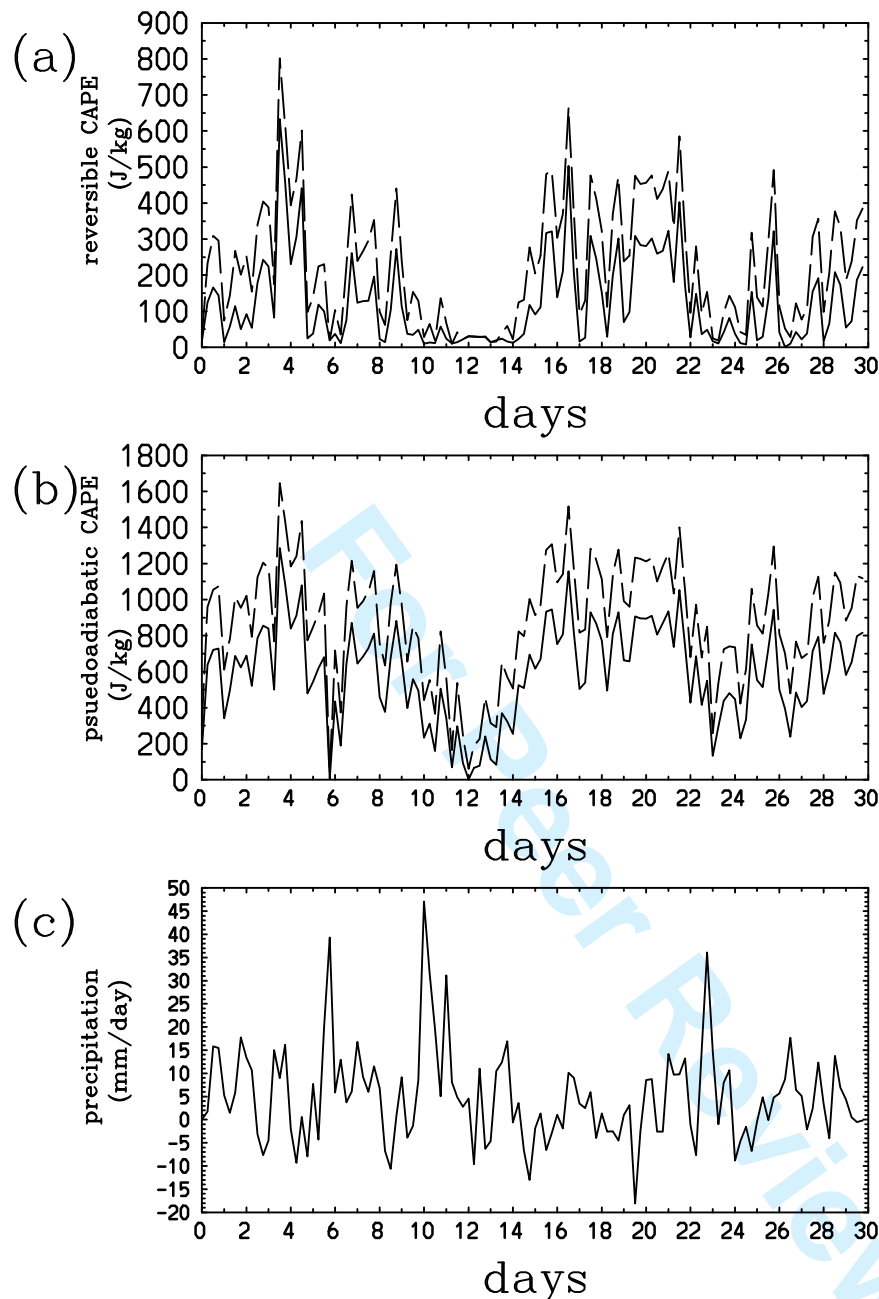
Fig. 3: Time series for the first 30 days of the TOGA–COARE period: (a) Reversible and (b) pseudo-adiabatic CAPE (J/kg), in which the solid curves show those based on the exact formula (12), the long–dash curves are with the standard approximation (3); (c) precipitation rate (mm/day). Note that the precipitation rate is indirectly estimated from the water-vapour budget, thus it is occasionally negative due to both the observation and diagnosis errors.
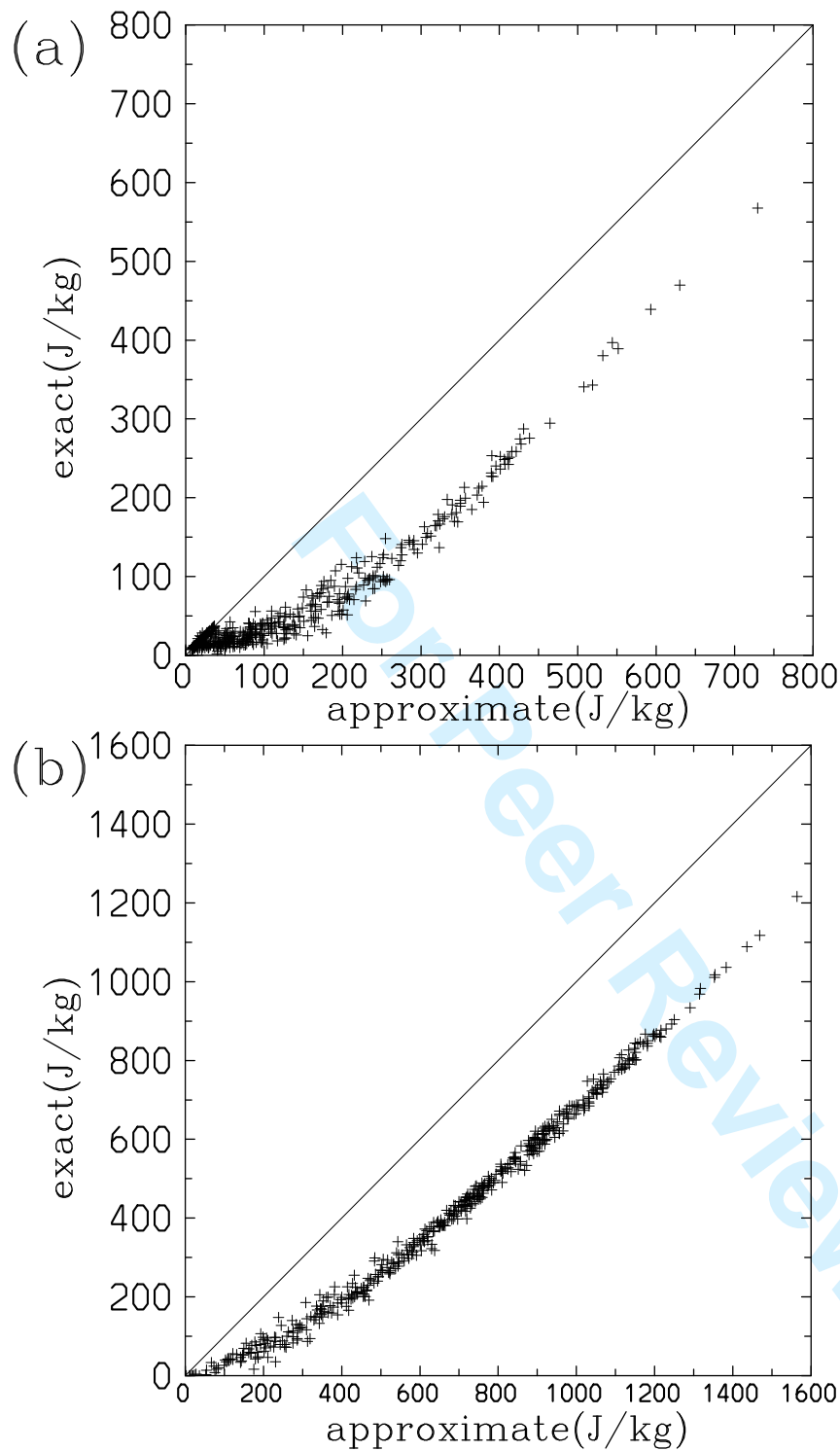
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Fig. 4: Scatter plots between the exact (vertical axis) and the approximate estimates (horizontal axis) of CAPE based on Eqs. (12) and (3), respectively for the whole TOGA–COARE period: (a) Reversible, (b) pseudo-adiabatic (J/kg).