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General relationships between pressure, weight and mass of a hydrostatic fluid

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In curved geometries the hydrostatic pressure in a fluid does not equal the weight per unit area of the fluid above it. General weight–pressure and mass–pressure relationships for hydrostatic fluids in any geometry are derived. As an example of the mass–pressure relationship, we find a geometric reduction in surface pressure as large as 5 mbar on Earth and 39 mbar on Titan. We also present a thermodynamic interpretation of the geometric correction which, as a corollary, provides an independent proof of the hydrostatic relationship for general geometries.

Keywords: hydrostatic pressure; fluid dynamics; atmospheric science

1. Introduction

The textbook definition of the hydrostatic pressure p_0 in terms of the weight W or the mass M of the fluid aloft, per unit surface area A_0, is

$$p_0 = \frac{W}{A_0} = \frac{Mg_0}{A_0}, \tag{1.1}$$

with g_0 the acceleration due to gravity. However, for curved geometries this relationship is not valid anymore, something that was pointed out by Newton (1726) in proposition 20 of book 2 of his *Principia*. Modified versions of equation (1.1) have been used to estimate the mass of the Earth's atmosphere by measuring the surface pressure. Trenberth (1981) provides such an estimate as well as a historic overview of earlier attempts. Bernhardt (1991*a*,*b*) provides an analysis of all contributions to the surface pressure on Earth, including the geometric effect. In a didactic article, Bannon *et al.* (1997), hereafter BBK, explicitly isolate the effect of geometry on the relationship between surface pressure and weight of the atmosphere and give a compact formula with the correction to the weight–pressure relationship expressed as a functional of the vertical pressure profile. They show how for spherical and cylindrical surfaces it is found that the pressure is lower because in these geometries lateral pressure forces will have an upward component, helping to hold up the air column.

We present here a more general version of BBK's weight–pressure relationships and also derive the general mass–pressure relationship valid for any geometry. We also introduce a thermodynamic interpretation of the

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geometric corrections to these relationships. In this interpretation the lateral (perpendicular to the vertical) pressure forces act as an effective negative surface tension which contributes a negative capillary pressure to the surface pressure budget. The thermodynamic interpretation serves as an independent derivation of hydrostatic balance in general geometries.

2. Hydrostatic pressure and fluid weight

The geopotential Φ is related to the local acceleration due to gravity by

$$\boldsymbol{g} = -\nabla \Phi. \tag{2.1}$$

We can define the 'vertical' as the direction parallel to the gradient of the geopotential Φ and use the geopotential as a vertical coordinate. The hydrostatic pressure p in this vertical coordinate is defined by

$$\frac{\mathrm{d}p}{\mathrm{d}\Phi} = -\rho, \tag{2.2}$$

with both the pressure p and the density ρ as functions of the geopotential only

$$p = p(\Phi), \qquad \rho = \rho(\Phi). \tag{2.3}$$

Without resorting to the equations of motion in a coordinate-free form, it is hard to see why the hydrostatic relationship in equation (2.2) should be valid for all geometries. Most informal derivations use variants of equation (1.1) which, however, is not valid in general geometries. Newton (1726) provides a general derivation in his *Principia*. Section 4 provides an independent derivation based on thermodynamic arguments alone.

If the gravitating mass density is finite, the range of the geopotential is finite too and it is often defined with a zero value at infinity and a finite negative value at the surface of the gravitating body. In meteorological applications, the geopotential at this surface is usually chosen as zero and increasing away from the surface to some maximum value $\Phi_{\max}$. Using this convention, the geopotential close to the surface equals gz with z, the geometric height above the surface.

The geometry of the geopotential field can be characterized by a function $V(\Phi)$ giving the geometric volume V inside a geopotential surface Φ, i.e. the volume containing geopotentials lower than Φ. For infinitesimal displacements $\mathrm{d}\Phi$, the volume between geopotential surfaces Φ and $\Phi+\mathrm{d}\Phi$ is $(\mathrm{d}V/\mathrm{d}\Phi)\mathrm{d}\Phi$. The volume derivative $\mathrm{d}V/\mathrm{d}\Phi$ can be related to the area of the enclosing geopotential surface by

$$\frac{\mathrm{d}V}{\mathrm{d}\Phi}\mathrm{d}\Phi = A\langle \mathrm{d}n\rangle, \tag{2.4}$$

with A the area of the geopotential surface and $\langle \mathrm{d}n\rangle$ the surface-averaged geometric thickness of the geopotential layer. We can now remap the geopotential onto a 'height' field $z(\Phi)$ by defining $\mathrm{d}z = \langle \mathrm{d}n\rangle$. With this definition of z, we can write

$$\mathrm{d}V = A\,\mathrm{d}z. \tag{2.5}$$

Because locally $\mathrm{d}\Phi = (\partial\Phi/\partial n)\mathrm{d}n = g\,\mathrm{d}n$, we find

$$\mathrm{d}z = \langle \mathrm{d}n\rangle = \langle g^{-1}\rangle \mathrm{d}\Phi. \tag{2.6}$$

For symmetric geopotential fields, where g is constant on geopotential surfaces, this new height coordinate is identical to the geometric height. For asymmetric geopotential fields, $z(\Phi)$ is the average geometric height of the geopotential surface. Note that z is not the same as the 'geopotential height' in meteorological literature; geopotential height is defined by Φ/g_c for some constant value of g_c (e.g. Trenberth & Guillemot 1994). Note also that the top boundary height $z_{max} = z(\Phi_{max})$ is normally infinity, contrary to geopotential height.

The total weight W of the fluid above some reference surface Φ_0 equals

$$W = \int_{\Phi>\Phi_0} \rho g \,\mathrm{d}A\,\mathrm{d}n = \int_{\Phi>\Phi_0} \rho \,\mathrm{d}A\,\mathrm{d}\Phi = \int_{\Phi_0}^{\Phi_{max}} \rho A \,\mathrm{d}\Phi. \tag{2.7}$$

Substituting the hydrostatic equation (2.2) and changing the integration variables, we can now write

$$W = -\int_{z_0}^{z_{max}} A \frac{\mathrm{d}p}{\mathrm{d}z} \mathrm{d}z. \tag{2.8}$$

Partial integration now leads to the general result

$$p_0 + \frac{1}{A_0} \int_{z_0}^{z_{max}} p \frac{\mathrm{d}A}{\mathrm{d}z} \mathrm{d}z = \frac{W}{A_0}, \tag{2.9}$$

with p_0 the surface pressure at geopotential Φ_0. Thus, the pressure p_0 at a surface is not equal to the weight per unit area above the surface, as in equation (1.1); it needs to be corrected with a geometric factor if the areas of the geopotential surfaces change with height.

Newton (1726) argues in proposition 20 of book 2 of the *Principia* that the surface pressure holds up the weight of a cylinder with the same surface area and the same vertical structure as the fluid in the curved geometry. In modern notation he argues that

$$W_{cyl} \equiv \int_{\Phi>\Phi_0} \rho A_0 \,\mathrm{d}\Phi = p_0 A_0, \tag{2.10}$$

with W_{cyl} the weight of the fluid in the described cylinder, which should be contrasted with the definition of the total weight in equation (2.7). The equality of W_{cyl} to $p_0 A_0$ is an expression of equation (2.2) in integral form. With this notation we can rewrite equation (2.9) as

$$W_{cyl} + \int_{z_0}^{z_{max}} p \frac{\mathrm{d}A}{\mathrm{d}z} \mathrm{d}z = W \tag{2.11}$$

and Newton comments on the difference between W_{cyl} and W, see §5.

As an example, the surface pressure of an atmosphere around a gravitating sphere can be easily calculated as the geopotential surfaces are symmetric if the sphere does not rotate. We therefore can set $z = r$ with r the radial coordinate on the sphere. In this coordinate, $A(r) = 4\pi r^2$ and we find

$$p_0 + \int_{r_0}^{r_{max}} \frac{2pr}{r_0^2} \mathrm{d}r = \frac{W}{A_0} \quad \text{(sphere)}. \tag{2.12}$$

This is the same as BBK's eqn (22). With the integral contribution positively definite, we find that the surface pressure is less than the weight per unit area of the atmosphere above it. BBK interpret this as the sideways pressure on a radial slice of the atmosphere having an upward component. In §4, we give a thermodynamic interpretation in terms of surface tension and capillary pressure.

BBK also discuss a hypothetical cylindrical world with an atmosphere around a gravitating cylinder. Again introducing a radial coordinate r and using that for a cylindrical world $A = 2\pi r H$ with H the axial height of the cylinder, we find that

$$p_0 + \int_{r_0}^{r_{\max}} \frac{p}{r_0} \mathrm{d}r = \frac{W}{A_0} \quad \text{(cylinder)}. \tag{2.13}$$

This is the same as BBK's eqn (20).

For a flat Earth with an atmosphere on top of a gravitating infinite plane, the area of the geopotential surfaces is not a function of the height of the surface. So for a flat Earth, we find that the hydrostatic pressure equals the weight of the atmosphere above it, as in equation (1.1).

A variant of equation (2.13) with the opposite sign of the integral term is valid in a quickly rotating centrifuge where the centrifugal force dominates the background gravity force. The height coordinate z is increasing inward into the centrifuge, against the centrifugal acceleration $g = \omega^2(r_0 - z)$. For a liquid of constant density ρ, it can be shown that at the surface ($z = 0$)

$$p_0 \left(1 - \frac{\alpha}{3} \frac{3 - 2\alpha}{2 - \alpha}\right) = \frac{W}{A_0}, \tag{2.14}$$

with $\alpha = D/r_0$ for D the radial depth of the fluid. For small $\alpha \ll 1$ the geometric factor is proportional to α, indicating a vanishing geometric correction for thinner layers of fluid; for a completely filled cylinder, $\alpha = 1$, the geometric correction increases the surface pressure by 50% compared with the weight per unit area. It can also be shown that the surface pressure in the centrifuge can be written as

$$p_0 = \frac{M\omega^2 r_0}{A_0}, \tag{2.15}$$

that is, the pressure is the same as the centrifugal force of a mass M at radius r_0 divided by the cylinder area at radius r_0. Interestingly, there is no difference in the surface pressure between liquids of different densities at the same total mass, even though a denser liquid will have more of its bulk located at higher centrifugal accelerations. In §3, this is clarified in a more general context.

3. Hydrostatic pressure and fluid mass

Equation (2.9) does not completely isolate the effects of geometry on the surface pressure because the weight is also dependent on the gravity field and therefore the geometry. We can perform a similar calculation as in §2 by considering the mass M rather than the weight W outside a geopotential surface Φ_0. Using the same vertical coordinate $z(\Phi)$ as before, the mass M is defined by

$$M = \int_{z_0}^{z_{\max}} \rho A \, \mathrm{d}z. \tag{3.1}$$

Substituting the hydrostatic equation (2.2) and performing a partial integration, it can be shown that

$$p_0 + \frac{1}{A_0 \langle g_0^{-1} \rangle} \int_{z_0}^{z_{\max}} p \frac{\mathrm{d}}{\mathrm{d}z} (A \langle g^{-1} \rangle) \mathrm{d}z = \frac{M}{A_0 \langle g_0^{-1} \rangle}, \tag{3.2}$$

where we have used equation (2.6) to transform between geopotential and z coordinates. This mass–pressure relationship is more complex than the weight–pressure relationship, equation (2.9), but it does completely isolate the geometric contribution to the surface pressure.

For the centrifuge we can show that $A\langle g^{-1}\rangle$ is independent of the height coordinate—both A and g are linear in r—so that the geometric contribution vanishes and the above equation reduces to equation (2.15).

We can cast equation (3.2) in purely geometric terms if we assume that the mass of the hydrostatic fluid can be ignored compared with that of the gravitating body, so we exclude self-gravitating atmospheres and we ignore contributions of any rotation to the geopotential. In this case the gravitational potential in terms of the mass density ρ_g of the gravitating body is

$$\nabla^2 \Phi = 4\pi G \rho_g, \tag{3.3}$$

with G the gravitational constant. Using Gauss' theorem we then find

$$\langle g \rangle A = 4\pi G \int \rho_g = \text{const.} \tag{3.4}$$

The constant can be rewritten using the values of g and A at the reference surface

$$\langle g \rangle A = \langle g_0 \rangle A_0. \tag{3.5}$$

This makes explicit that we cannot ignore variations in g while taking into account geometric effects and vice versa, something that had apparently not been appreciated in the original calculations for the Earth's atmosphere by Trenberth (1981). To first order in variations of the gravitational acceleration over a geopotential surface, we have $\langle g^{-1}\rangle = \langle g \rangle^{-1}$, which can be substituted in equation (3.2) to find

$$p_0 + \frac{1}{A_0^2} \int_{z_0}^{z_{\max}} p \frac{\mathrm{d}A^2}{\mathrm{d}z}\, \mathrm{d}z = \frac{M \langle g_0 \rangle}{A_0}. \tag{3.6}$$

The geometric contribution in this equation is different from that in equation (2.9) because we have now made explicit the geometric contribution to the weight as well.

We can make an estimate of the geometric corrections for Earth conditions, as was done in BBK for the weight–pressure relationship. We use an approximation of the Earth's pressure profile of $p = p_0 \exp(-z/H)$ with p_0 the mean surface pressure at the Earth's surface and H the pressure scale height (approx. 8 km). Assuming a spherical Earth, it can be shown that equation (3.6) becomes

$$p_0(1 + 4\epsilon(1 + 3\epsilon(1 + 2\epsilon(1 + \epsilon)))) = \frac{M g_0}{A_0}, \tag{3.7}$$

with $\epsilon = H/a$ and a the Earth's mean radius. We find that the geometric correction changes the flat-Earth mass–pressure relationship by approximately 0.5%. So for Earth conditions, the geometric effect reduces the surface pressure by approximately 5 mbar globally. A similar value was found by Bernhardt (1991*b*) using different approximations.

Fulchignoni *et al.* (2005) present profile data for Titan's atmosphere as measured by the Huygens probe in 2004. From their results it can be derived that the pressure scale height in the lowest 75 km can be estimated to be 16.7 km. With Titan's radius of 2575 km, we find a geometric correction of 2.7%, corresponding to a geometric surface pressure reduction of 39 mbar.

4. Thermodynamic interpretation

The weight–pressure relationship, equation (2.9), has a thermodynamical interpretation, which sheds light on the physics involved and provides an independent derivation of the hydrostatic relation, equation (2.2). We consider the work done by lifting the fluid layer against gravity. On a flat Earth this work would equal $p_0 A_0 \delta z$ when lifting over an infinitesimal amount δz. This work should equal the increase in potential energy of the fluid, which is $W\delta z$ with W the weight of the fluid. Equating these two, we get the familiar weight–pressure relationship, equation (1.1).

However, in a curved geometry, the fluid will not only change its centre of gravity but also will expand laterally. This will also be part of the energy budget. To quantify this, we will lift the fluid by an amount δz, where the vertical coordinate is as before a remapping of the geopotential field, $z = z(\Phi)$. We can think of the fluid as made up of shells of thickness $\mathrm{d}z$, area A and volume $A\,\mathrm{d}z$. As before, the pressure work required to lift the fluid vertically over δz equals $p_0 A_0 \delta z$, and the increase in potential energy equals $W\delta z$. Furthermore, each shell will expand its area by an amount $\delta A = (\mathrm{d}A/\mathrm{d}z)\delta z$, thus increasing its volume by $\delta V = \delta A\,\mathrm{d}z$. This volume increase requires an amount of work $-p\delta V = -p(\mathrm{d}A/\mathrm{d}z)\delta z$. This last contribution has to be integrated over all shells above the reference level. We now find the energy budget

$$p_0 A_0 \delta z = W\delta z - \int_{z_0}^{z_{\max}} p\left(\frac{\mathrm{d}A}{\mathrm{d}z}\delta z\right)\mathrm{d}z, \tag{4.1}$$

which is equivalent to equation (2.9). Note that we have not used the hydrostatic equation, equation (2.2), here: the equivalence of the above equation and equation (2.9) is a proof of the hydrostatic relationship under the imposed conditions, namely that all fields are functions of geopotential only and that there are no other forms of energy conversion involved. For example, vertical accelerations would involve conversions to kinetic energy but, as is well known, would at the same time invalidate the hydrostatic relationship.

A related way of interpreting the contribution of each shell to the surface pressure would be to represent the lateral forces on the expansion of the fluid shells by an effective 'surface tension' γ defined such that the work required to change the area of the fluid shell equals $\gamma\delta A$ (e.g. Adkins 1984). We therefore find that for each fluid shell,

$$\gamma = -p\,\mathrm{d}z. \tag{4.2}$$

The effective surface tension of the shell is negative as an increase in area releases energy. Each shell under surface tension will represent a capillary pressure jump across the shell

$$\mathrm{d}p = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{4.3}$$

with R_1 and R_2 the local radii of curvature of the geopotential surface in two orthogonal directions (e.g. Batchelor 1967). For example, a spherical surface at radius r would represent a pressure jump of $\mathrm{d}p = 2\gamma/r$, corresponding to a downward force of $\mathrm{d}F = A\,\mathrm{d}p = 8\pi\gamma r$. Integrating this force over all shells and dividing the total downward force by the surface area $A_0 = 4\pi r_0^2$, we find the

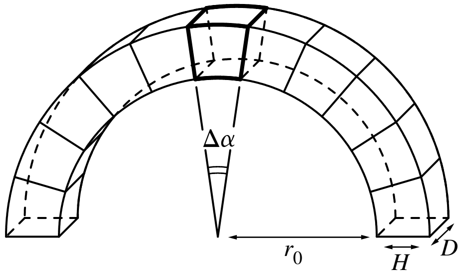

Figure 1. Geometry of masonry arch with keystone highlighted. The keystone has weight W and experiences a lateral force F on either side.

geometric contribution to the surface pressure in equation (2.12), which can now be interpreted as the integral effect of 'capillary' pressure contributions from different geopotential shells.

For general geometries, the pressure jump dp due to surface tension can be calculated from (see Adkins 1984)

$$\gamma\,\delta A = \mathrm{d}p\,\delta V. \tag{4.4}$$

In our hydrostatic fluid set-up, we have $\delta A/\delta V = (\mathrm{d}A/\mathrm{d}z)/A$, from which the total downward force dF of each fluid shell can be determined as $\mathrm{d}F = A\,\mathrm{d}p = \gamma(\mathrm{d}A/\mathrm{d}z)$. Again, integrating this force over all shells and dividing the total downward force by the surface area A_0, we find the geometric contribution to the surface pressure in equation (2.9).

5. Postscript

The idea that lateral pressures can help to keep up a weight against gravity was already noted by Newton as corollary 1 to proposition 20 in book 2 of the *Principia*.

> Therefore the bottom is pressed by the whole weight of the incumbent fluid, but sustains only that part which is described in this proposition, the rest of the weight being sustained by the vaulted shape of the fluid.
>
> (Newton 1726)

The vaulting effect is of great importance in the study of flowing granular media (e.g. Baxter *et al.* 1989; Jaeger & Nagel 1992) where it can lead to significant pressure drops and flow stagnation.

The vaulting effect is perhaps most striking in masonry arches that are used to support weight over wide spans. Figure 1 shows a schematic of an arch with the keystone highlighted. The weight W of the keystone is kept up by the vertical components of the lateral forces F on the sides of the keystone. A straightforward analysis of the geometry shows that

$$2F\sin(\Delta\alpha/2) = W. \tag{5.1}$$

However, equation (2.13) should be equally applicable to the geometric pressure contribution carrying the full weight of the keystone. Because gravity in the arch always points downward, rather than radially inward, we expect equation (2.13) to be valid only for small values of $\Delta\alpha$. With $A = Dr\Delta\alpha$ and $p = FDH$, we find from equation (2.13)

$$F\Delta\alpha \approx W, \tag{5.2}$$

which approaches the exact result for small $\Delta\alpha$. The masonry arch provides a didactic image to help understand geometric contributions to the pressure.

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