On the generalized eigenvalue problem for the Rossby wave vertical velocity in the presence of mean flow and topography

Rémi Tailleux *

University of Reading, Department of Meteorology

* *Corresponding author address:* University of Reading, Department of Meteorology, Earley Gate, PO Box 243, Reading RG6 6BB, United Kingdom

E-mail: R.G.J.Tailleux@reading.ac.uk

ABSTRACT

In a series of papers, Killworth and Blundell (2004,2005,2007) have proposed to study the effects of a background mean flow and topography on Rossby wave propagation by means of a generalized eigenvalue problem formulated in terms of the vertical velocity, obtained from a linearization of the primitive equations of motion. However, it has been known for a number of years that this eigenvalue problem contains an error, which Peter Killworth was prevented from correcting himself by his unfortunate passing, and whose correction is therefore taken up in this note. Here, we show in the context of quasi- geostrophic (QG) theory that the error can ultimately be traced to the fact that the eigenvalue problem for the vertical velocity is fundamentally a nonlinear one (the eigenvalue appears both in the numerator and denominator), unlike that for the pressure. The reason that this nonlinear term is lacking in Killworth and Blundell's theory comes from neglecting the depth-dependence of a depth-dependent term. This nonlinear term is shown on idealized examples to alter significantly the Rossby wave dispersion relation in the high-wavenumber regime, but is otherwise irrelevant in the long wave limit, in which case the eigenvalue problems for the vertical velocity and pressure are both linear. In the general dispersive case, however, one should first solve the generalized eigenvalue problem for the pressure vertical structure, and if needed, diagnose the vertical velocity vertical structure from the latter.

1. Introduction

A central question in the theory of ocean variability is how the barotropic and baroclinic normal modes of the standard linear theory (SLT), e.g. Gill (1982), are modified by the presence of a background mean flow and variable topography? To address this issue, progress over the past decades has principally come from investigating the nature of the solutions of the equations of motion linearized around a background mean flow, most often in the context of quasi-geostrophic (QG) theory. Under the WKB approximation, which assumes that the scales over which the background mean flow and topography vary are large compared to that of the waves considered, approximately separable wave solutions still exist, whose vertical structure can be obtained as the eigenmodes of a non-self adjoint eigenvalue problem. When the problem is formulated by linearizing the quasi-geostrophic potential vorticity evolution equation around a background zonal mean flow for instance, the generalised eigenvalue problem thus obtained is generally naturally formulated in terms of the vertical structure for the pressure, e.g., see Fu and Chelton (2001) and Aoki et al. (2009) for recent examples.

In the classical SLT, i.e., in absence of mean flow and topography, it has long been known that the eigenvalue problem defining the standard barotropic and baroclinic modes can be indifferently formulated in terms of the vertical structure for either the pressure or vertical velocity. Yet, we were unable to find any published derivation of the generalised eigenvalue problem (i.e., accounting for mean flow and topography) for the vertical structure of the vertical velocity in the context of QG theory. Motivated by the fact that boundary conditions are generally simpler for the vertical velocity, Killworth and Blundell (2004, 2005, 2007) (KB04,KB05,KB07 thereafter) sought to formulate a generalized eigenvalue problem for the vertical structure of the vertical velocity from directly linearizing the primitive equations. Using QG scaling, they first obtained linearized equations for the horizontal velocity components in terms of Welander (1959)'s M function, which once inserted into the continuity equation led to the following expression for the vertical derivative of w:

$$w_z = \frac{ikM_z}{2\Omega a^2 \sin^2\theta} - \frac{iRM_z}{a^2 f^2}\left(\frac{k^2}{\cos^2\theta} + l^2\right) + \text{small} \tag{1}$$

where $R = \mathbf{k}_{dim} \cdot \overline{\mathbf{u}} - \omega$ is minus the Doppler-shifted frequency ω, $\overline{u}$ is the zonal background mean flow, $\mathbf{k}_{dim} = (k_x, k_y)$ is the dimensional wave vector, (k, l) are the angular zonal and meridional wavenumbers, which are related to the dimensional wavenumbers (k_x, k_y) by $k_x = k/(a\cos\theta)$ and $l = k_y/a$, a is the Earth radius, θ is the latitude, Ω is Earth's rotation rate, and $f = 2\Omega\sin\theta$ is the local Coriolis parameter. At this point, KB04 sought to derive an expression for w by vertically integrating Eq. (1), which they took to be given by:

$$w = \frac{ikM}{2\Omega a^2 \sin^2\theta} - \frac{iRM}{a^2 f^2}\left(\frac{k^2}{\cos^2\theta} + l^2\right). \tag{2}$$

Such a derivation is valid, however, only if R can be assumed to be independent of z, but as noted earlier, R is given by:

$$R = \mathbf{k}_{dim} \cdot \overline{\mathbf{u}} - \omega \tag{3}$$

and in general will depend on z because the background mean flow depends on z.

The above error was first identified by Roger Samelson (Samelson, 2007, personal communication) who had pointed it out to Peter Killworth at the time. But despite his best efforts, Peter Killworth passed away before he could find a cure to the problem. The main objective of this paper is twofold: 1) to clarify the nature of the error and show how to redress it; 2) understand how the error affects some of KB04's conclusions regarding the nature of the dispersion relation of Rossby waves in presence of a background mean flow and topography. The issue is important to clarify, because it also affects KB05's results, KB07's discussion of forced modes and baroclinic instability, and is expected to alter some of the conclusions of the comparison of KB04's theoretical dispersion relations against observations recently carried out by Maharaj et al. (2007, 2009).

In this paper, we note that KB04's theory appears to rely on the same scaling arguments as those underlying the construction of QG theory, and hence seek to derive a generalised eigenvalue problem for the vertical velocity directly from QG theory, which is done in Section 2. The main result is that the eigenproblem thus obtained only differ from that of KB04 by a term that makes the QG eigenvalue problem for the vertical velocity nonlinear, and hence intractable. Section 3 illustrates on some idealised examples that such a term affects KB04's dispersion relations mostly in the high wave numbers regime. Section 4 summarizes the results and discusses some of its consequences.

2. Generalized eigenvalue problem for w in QG theory

The standard starting point for generalising the SLT to account for the effects of a background mean flow and topography is the QG evolution equation for the potential vorticity (PV) equation:

$$\frac{D_g q}{Dt} = 0, \tag{4}$$

where $D_g/Dt = \partial_t + J(\Psi, \cdot)$ is advection by the geostrophic velocity $(u_g, v_g) = -\Psi_y, \Psi_x$, with Ψ being the geostrophic streamfunction, and q is the potential vorticity:

$$q = f_0 + \beta y + \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \Psi}{\partial z}\right). \tag{5}$$

The geostrophic stream function Ψ is related to the buoyancy b, pressure p, and vertical velocity w through the following relations:

$$b = f_0 \frac{\partial \Psi}{\partial z}, \qquad p = \rho_0 f_0 \Psi, \tag{6}$$

$$w = -\frac{D_g}{Dt}\left(\frac{f_0}{N^2}\frac{\partial \Psi}{\partial z}\right), \tag{7}$$

e.g., Vallis (2007). As shown by several authors, e.g., Fu and Chelton (2001), Aoki et al. (2009), the QG evolution equation linearized around a zonal background mean flow $\overline{u} = \overline{u}(z)$ [neglecting $\overline{u}_{yy}$ relative to β] admits separable wave-like solutions $\Psi \propto F(z)e^{i(k_x x + k_y y - \omega t)}$, whose vertical structure F can be regarded as the eigenmodes of the following eigenvalueproblem:

$$(\overline{u} - c)\left[\frac{d}{dz}\left(\frac{f_0^2}{N^2}\frac{dF}{dz}\right) - K^2 F\right] + \left[\beta - \frac{d}{dz}\left(\frac{f_0^2}{N^2}\frac{d\overline{u}}{dz}\right)\right] F = 0, \tag{8}$$

where $K^2 = k_x^2 + k_y^2$, with suitable boundary conditions whose precise form depends on whether a variable or flat bottom topography is considered, where $c = \omega/k_x$ is the zonal phase speed.

Our objective is to obtain the corresponding eigenvalue problem for the vertical velocity w. To that end, linearizing Eq. (7) yields the following linearized expression for w:

$$w = -\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\frac{f_0}{N^2}\frac{\partial \Psi'}{\partial z} + \frac{f_0}{N^2}\frac{d\overline{u}}{dz}\frac{\partial \Psi'}{\partial x}. \tag{9}$$

This implies for the vertical structure of wave-like solutions $w = W(z)e^{i(k_x x + k_y y - \omega t)}$ that it is related to the pressure vertical structure F through:

$$W = -if_0 k_x \left[\frac{(\overline{u} - c)}{N^2} \frac{dF}{dz} - \frac{1}{N^2} \frac{d\overline{u}}{dz} F \right]. \tag{10}$$

In order to arrive at the eigenvalue problem satisfied by W, we successively differentiate the latter expression with respect to z, by making use of the links between F and W to successively eliminate terms involving F and its derivatives. Thus, differentiating the latter equation a first time yields:

$$\frac{dW}{dz} = -if_0 k_x \left\{ (\overline{u} - c) \frac{d}{dz} \left(\frac{1}{N^2} \frac{dF}{dz} \right) - \frac{d}{dz} \left(\frac{1}{N^2} \frac{d\overline{u}}{dz} \right) F \right\} \tag{11}$$

By taking advantage of the fact that F satisfies the eigenvalue problem Eq. (8), it is possible to remove the second-order term in F to simplify this expression as follows:

$$\frac{dW}{dz} = -\frac{ik_x}{f_0} \left\{ (\overline{u} - c) K^2 - \beta \right\} F = \frac{ik_x}{f_0} \left\{ \beta - K^2 (\overline{u} - c) \right\} F, \tag{12}$$

in which all first and second derivatives in F have been eliminated. Taking the vertical derivative a second time yields this time,

$$\frac{d^2 W}{dz^2} = \frac{ik_x}{f_0} \left[\beta - K^2 (\overline{u} - c) \right] \frac{dF}{dz} - \frac{ik_x K^2}{f_0} \frac{d\overline{u}}{dz} F. \tag{13}$$

By using Eq. (10), it is possible to express dF/dz in terms of F and W as follows:

$$\frac{dF}{dz} = \frac{1}{\overline{u} - c} \frac{\partial \overline{u}}{\partial z} F + \frac{iN^2}{f_0 k_x (\overline{u} - c)} W, \tag{14}$$

and then, by using Eq. (12), it is possible to express F in terms of dW/dz. As a result, the following eigenproblem is obtained:

$$(\overline{u} - c) \frac{d^2 W}{dz^2} - \left[1 - \frac{K^2 (\overline{u} - c)}{\beta - K^2 (\overline{u} - c)} \right] \frac{d\overline{u}}{dz} \frac{dW}{dz} + \frac{[\beta - K^2 (\overline{u} - c)] N^2}{f_0^2} W = 0 \tag{15}$$

The surprising result here is that Eq. (15) is a nonlinear eigenvalue problem because of the term $K^2(\overline{u} - c)/[\beta - K^2(\overline{u} - c)]$, which involves the eigenvalue c both in the numerator and denominator. In contrast, the eigenvalue problem derived by KB04 (in absence of meridional mean flow) is given by:

$$(\overline{u}-c)\,\frac{d^2W_{KB}}{dz^2}-\frac{d\overline{u}}{dz}\frac{dW_{KB}}{dz}+\frac{\left[\beta-(\overline{u}-c)K^2\right]N^2}{f_0^2}W_{KB}=0. \quad (16)$$

The two eigenproblems are identical but for the missing nonlinear term in Eq. (16). In the long wave limit studied by many authors, e.g., P. D. Killworth and de Szoeke (1997); Tailleux (2004); de Verdière and Tailleux (2005), however, the nonlinear term vanishes and the eigenvalue problems for the pressure and vertical velocity are equally simple and linear. In some other instances, such as in the case of the internal waves in the traditional f-plane approximation, it is in terms of the vertical velocity that the problem is most conveniently formulated, as it is the problem in terms of pressure that becomes nonlinear, e.g., Gill (1982) (Eq. 8.4.10).

3. Particular example of the differences

As mentioned above, the nonlinear term $K^2(\overline{u}-c)/[\beta-K^2(\overline{u}-c)]$ responsible for the difference between the two eigenproblems given by Eqs. (15) and (16) clearly vanishes in the long wave limit $K\to 0$, so that we expect it to affect the eigensolutions only at large wave numbers. That this is indeed the case is illustrated here in the particular case of the following idealized mean flow and stratification:

$$\overline{u}(z)=u_{min}+\Delta u_0\exp\left\{(z-z_0)^2/\delta^2\right\}, \quad (17)$$

$$N^2(z)=N_0+\Delta N\exp\left\{(z-z_0)^2/\delta^2\right\}, \quad (18)$$

which are displayed in Fig. 1. The consideration of an idealized example is sufficient for the present purposes of illustrating that the error made in KB's papers is not innocuous. A more complete investigation of the consequences of the error made in KB04 on the conclusions of all the papers that rely on KB's papers is beyond the scope of this paper. The dispersion relations for the QG and KB eigenproblems were computed by solving the discretized versions of the QG eigenvalue problem for pressure and KB04 eigenproblem for the vertical velocity by using Matlab's standard eigenvalue routines. In order to compare the vertical velocity in each theory, the vertical velocity modal structure was diagnosed using the pressure modal structure by using the discretized version of Eq. (10).

In absence of a background mean flow, both QG and KB's theories should be strictly equivalent. This is found to be the case, as illustrated in the left panels of Figs. 2 and 3, corresponding respectively to the use of the standard flat-bottom boundary condition and that of bottom-pressure compensation theory of Tailleux and McWilliams (2001), which can be regarded as a limiting case of the effect of bottom topography in the infinitely-steep slope limit, e.g., Tailleux (2003). As discussed above, we expect the two theories to yield dissimilar results in presence of mean flow primarily at large wavenumbers. This is illustrated in the right panels of Figs. 2 and 3, which show that for wavenumbers larger than the Rossby radius of deformation, the two theories may start to differ dramatically, demonstrating the importance of the corrective term overlooked by KB04 in such a region of the wavenumbers space, both for a flat bottom and the bottom pressure compensation boundary conditions. Note that in the asymptotic limit $k_x \to -\infty$, the dispersion relationship becomes quasi-nondispersive, and given by $\omega = -u_{min} k_x$, where u_{min} is the absolute minimum value of $\overline{u}(z)$ in the vertical, as demonstrated by Gnevyshev and Shrira (1989). Although KB's dispersion relationship also appears to be quasi-nondispersive at high wavenumbers, its behavior appears nevertheless quite different from that of the QG case. Moreover, we failed to obtain an analytical result for the asymptotic behavior of KB's dispersion relationship for $k_x \to -\infty$.

4. Summary and conclusions

In this paper, we derived the generalised QG eigenvalue problem for the vertical velocity normal mode structure of Rossby waves in presence of a background mean flow and topography. Previously, such an eigenproblem had been formulated only for the pressure. Surprisingly, the vertical velocity eigenproblem appears to be nonlinear (the eigenvalue appears both in the numerator and denominator), which is very uncommon and not easily anticipated given that the eigenproblem for the pressure is linear. Such a result shows how KB04's derivation might be corrected; at the same time, it also shows that the actual eigenproblem for the vertical velocity is not easily tractable, and that the investigation of the propagation properties of Rossby waves in presence of mean flow and topography is more easily addressed by solving the pressure eigenvalue problem. Whenever the vertical velocity structure W is needed, it is most conveniently diagnosed a posteriori from the knowledge of the pressure vertical structure F by using (the discretized version of) Eq. (10).

The results also show that the error made by KB04 is in fact equivalent to neglecting the term making the QG eigenproblem for the vertical velocity nonlinear. Such a term is small in the long wave limit $K \to 0$, so that the error is mostly of consequence for understanding the behaviour of the Rossby wave dispersion relation at high wave numbers. The study of the latter regime is an old problem, which was investigated in significant details by Gnevyshev and Shrira (1989), and is therefore not discussed in more details here.

Acknowledgments.

The author gratefully acknowledges financial support from CNES, and Dudley Chelton and Roger Samelson for encouragements in pursuing this study. This paper is dedicated to the memory of Peter Killworth, who was a constant source of inspiration in the study of oceanic Rossby waves. We hope that he would have agreed on the solution proposed.

REFERENCES

Aoki, K., A. Kubokawa, H. Sasaki, and Y. Sasai, 2009: Midlatitude Baroclinic Rossby Waves in a High-Resolution OGCM Simulation. *J. Phys. Oceanogr.*, **39**, 2264–2279, doi: {10.1175/2009JPO4137.1}.

de Verdière, A. C. and R. Tailleux, 2005: The interaction of a baroclinic can flow with long rossby waves. *J. Phys. Oceanogr.*, **35**, 865–879.

Fu, L. and D. B. Chelton, 2001: *Large-scale ocean circulation*, Vol. Satellite Altimetry and Earth Sciences, 69. Academic Press, 133-169 pp.

Gill, A. E., 1982: *Atmosphere-Ocean Dynamics*. Academic Press, 662 pp.

Gnevyshev, V. and V. Shrira, 1989: Monochromatic rossby wave transformation in zonal flow critical layer. *Izv. Acad. Sci. USSR, Atmos. Oceanic Phys.*, **25**, 628–635.

Killworth, P. and J. Blundell, 2004: The dispersion relation for planetary waves in the presence of mean flow and topography. Part I: Analytical theory and one-dimensional examples. *J. Phys. Oceanogr.*, **34**, 2692–2711, doi:{10.1175/JPO2635.1}.

Killworth, P. and J. Blundell, 2005: The dispersion relation for planetary waves in the presence of mean flow and topography. Part II: Two-dimensional examples and global results. *J. Phys. Oceanogr.*, **35**, 2110–2133, doi:{10.1175/JPO2817.1}.

Killworth, P. D. and J. R. Blundell, 2007: Planetary wave response to surface forcing and instability in the presence of mean flow and topography. *J. Phys. Oceanogr.*, **37**, 1297–1320, doi:{10.1175/JPO3055.1}.

Maharaj, A. M., P. Cipollini, N. J. Holbrook, P. D. Killworth, and J. R. Blundell, 2007: An evaluation of the classical and extended Rossby wave theories in explaining spectral estimates of the first few baroclinic modes in the South Pacific Ocean. *Ocean Dynamics*, **57**, 173–187, doi:{10.1007/s10236-006-0099-5}.

Maharaj, A. M., N. J. Holbrook, and P. Cipollini, 2009: Multiple westward propagating signals in South Pacific sea level anomalies. *J. Geophys. Res. Oceans*, **114**, doi:{10.1029/2008JC004799}.

P. D. Killworth, D. B. C. and R. de Szoeke, 1997: The speed of observed and theoretical long extratropical planetary waves. *J. Phys. Oceanogr.*, **27**, 1946–1966.

Tailleux, R., 2003: Comments on "The effect of bottom topography on the speed of long extratropical planetary waves". *J. Phys. Oceanogr.*, **33**, 1536–1541, doi:{10.1175/1520-0485(2003)033⟨1536:COTEOB⟩2.0.CO;2}.

Tailleux, R., 2004: A wkb analysis of the surface signature and vertical structure of long extratropical baroclinic rossby waves over topography. *Ocean Modelling*, **6**, 191–219.

Tailleux, R. and J. McWilliams, 2001: The effect of bottom pressure decoupling on the speed of extratropical, baroclinic Rossby waves. *J. Phys. Oceanogr.*, **31**, 1461–1476, doi:{10.1175/1520-0485(2001)031⟨1461:TEOBPD⟩2.0.CO;2}.

Vallis, G., 2007: *Atmosphere Ocean Circulation*. Cambridge University Press, 700 pp.

Welander, P., 1959: An advective model of the ocean thermocline. *Tellus*, **XI**, 309–318.

List of Figures

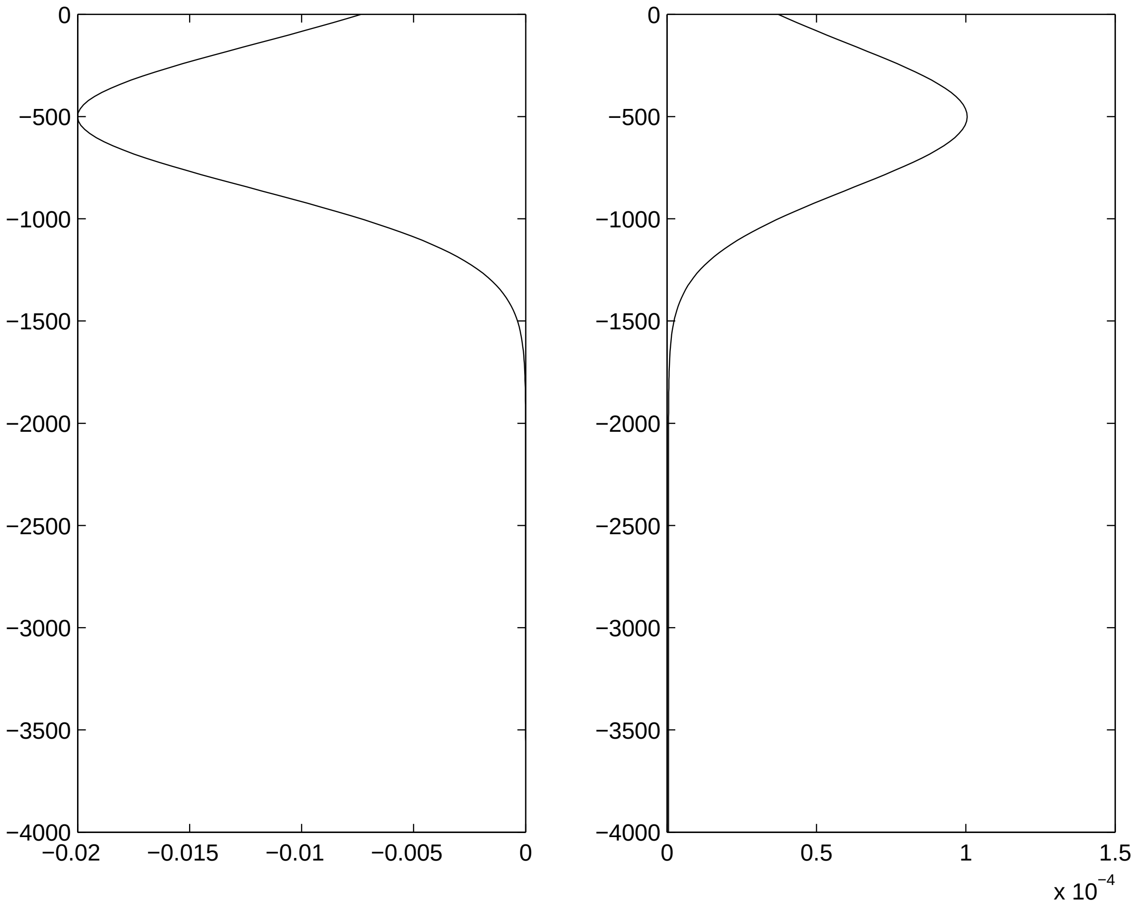

FIG. 1. Idealized profiles for the squared buoyancy frequency N^2 (right panel, in s^{-1}) and zonal velocity profile (left panel, in m/s).

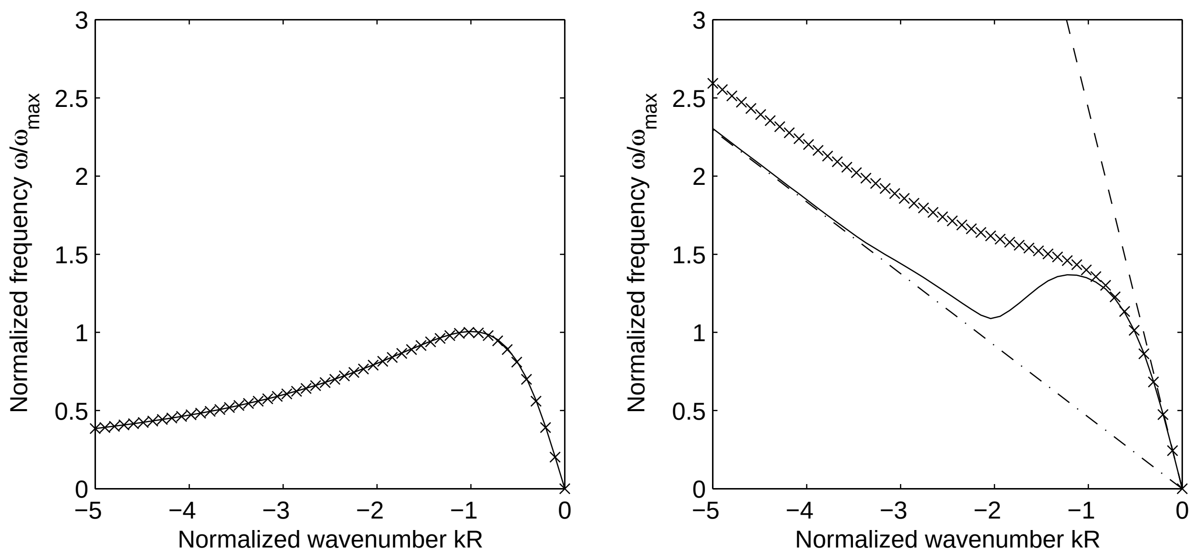

FIG. 2. Comparison of QG (solid line) and KB (crosses) flat-bottom theories. (Left panel) Dispersion relation in absence of mean flow as predicted by classical QG theory (solid line) and KB theory (crosses). (Right panel) Dispersion relation for Rossby waves affected by the idealized Gaussian mean flow illustrated in Fig. 1, as predicted by the classical QG theory (continuous line) and by KB theory (crosses). The dashed-dotted line represents the nondispersive dispersion relationship $\omega = -u_{min}k_x$, while the dashed line represents the nondispersive relationship tangent at $k_x = 0$, i.e., $\omega = c(k_x = 0)k$, where u_{min} is the absolute minimum of the horizontal zonal velocity along the vertical (which is strictly negative and located at $z = -500\,\text{m}$, according to Fig. 1). Frequency is normalized by the maximum frequency of the flat-bottom, no mean flow standard linear theory, while the zonal wavenumber is normalized by the inverse of the Rossby radius of deformation.

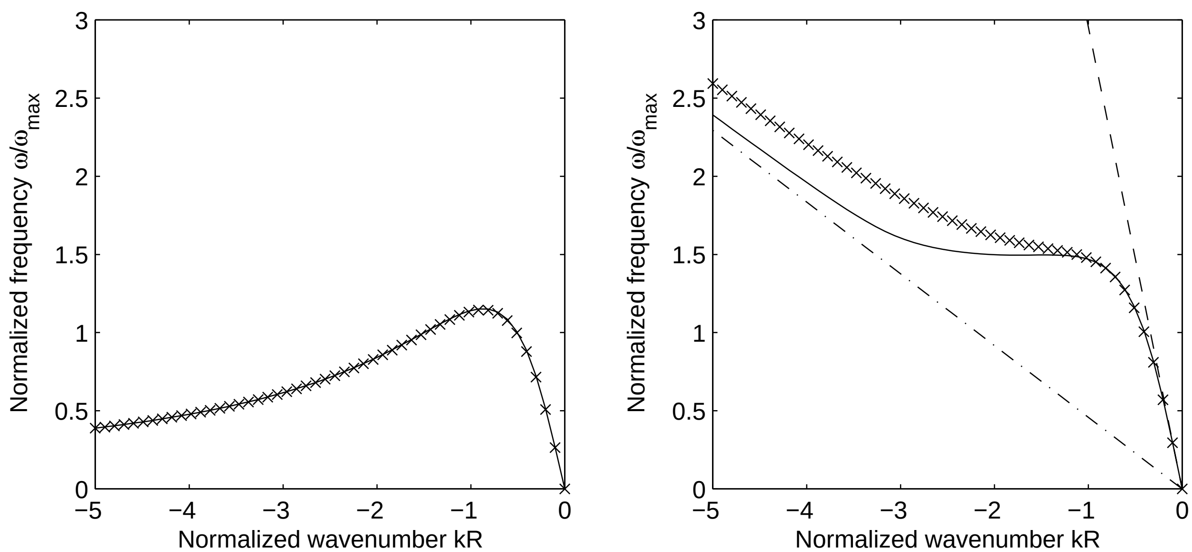

Fig. 3. Comparison of QG (solid line) and KB (crosses) theories using the bottom boundary condition of the bottom-pressure compensation (BPC) theory of Tailleux and McWilliams (2001). (Left panel) Dispersion relation in absence of mean flow as predicted by classical QG theory (solid line) and KB theory (crosses). (Right panel) Dispersion relation for Rossby waves affected by the idealized Gaussian mean flow illustrated in Fig. 1 as predicted by the classical QG theory (continuous line) and by KB theory (crosses). The dashed-dotted line represents the nondispersive dispersion relationship $\omega = -u_{min}k_x$, as in Fig. 2, while the dashed line represents the nondispersive relationship tangent at $k_x = 0$, i.e., $\omega = c(k_x = 0)k_x$. Frequency is normalized by the maximum frequency of the flat-bottom, no mean flow theory, while the zonal wavenumber is normalized by the inverse of the Rossby radius of deformation.