Identifying and quantifying the spurious sources and sinks of energy in hydrostatic Boussinesq primitive equation models used for climate studies

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SUMMARY

This paper presents an attempt at identifying and quantifying theoretically the spurious sources and sinks of energy associated with the hydrostatic Boussinesq primitive equations that form the basis for most ocean general circulation models (OGCMs) presently used for climate studies. At least four main sources of errors are found that prevent such models to possess a well-defined closed energy budget and local energy balances. They arise from, by decreasing order of importance: 1) the kinetic energy dissipation term that in reality would be converted into heat, but which is “hanging” in OGCMs; 2) assuming potential temperature to be a conservative quantity; 3) the interaction of the Boussinesq approximation with the parameterizations of turbulent mixing of temperature and salinity; 4) some adiabatic compressibility effects due to the Boussinesq approximation. In principle, a fifth source of error in the form of spurious numerical energy sources/sinks must also exist in OGCMs.

Apart from error 1), most of the spurious energy sources/sinks are not sign definite, and therefore should not affect the volume-integrated energy budget significantly. Of main concern, however, are the local energy balances which are qualitatively different from that of the actual equations of motion, owing mostly to the conversion term between kinetic energy and internal energy being transferred from the conservation equation for kinetic energy from the conservation equation for gravitational potential energy; furthermore, these spurious energy sources/sinks could possibly be of the same order of magnitude as actual energy conversion terms occurring in the oceans. Although the actual impact of these errors on the overall accuracy and physical realism of the oceans is difficult to ascertain, it is speculated that it could impact on transient simulations, and on the transition toward different circulation regimes that require a significant reorganization of the different energy reservoirs.

Some possible solutions for improvement are examined. It is thus found that error 2) can be reduced by about one order of magnitude by using conservative temperature instead of potential temperature. On the other hand, using the anelastic approximation, which was initially thought as a possible way to greatly improve the accuracy of the energy budget, would only marginally improve error 4) with no impact on errors 1), 2) and 3).

KEYWORDS: Long term climate integration  Model uncertainties  Numerical errors

1. INTRODUCTION

The development of fast, accurate, and reliable numerical models of the climate system is a crucial part of understanding and predicting the magnitude and distribution of future global climate variations occurring in response to natural variability and human-induced greenhouse gases emissions. Ideally, one would like such models to be consistent with all known physical laws governing the various physical, biological, and chemical components of the climate system, but especially with the first and second laws of thermodynamics, as well as mass conservation, which are among the most important of all. It is widely held, however, that in any practical numerical implementation, the climate equations need first to be simplified and approximated to make them computationally tractable; unfortunately, it has proven difficult so far, if not impossible, to do so while remaining entirely physically consistent.

The hydrostatic Boussinesq approximation constitutes one particular example that is widely used in most current ocean general circulation models. It was originally introduced by Boussinesq (1903), while Spiegel and Veronis (1969) are generally credited for the first rigorous scaling analysis of the equations. The approximation has been since analysed by many other authors, and extensively discussed in the literature, often in connection with the anelastic approximation (Ogura and Phillips, 1962) of which it can be regarded as a particular case, e.g., Lilly (1996), Bannon (1996). As a...
consequence of the Boussinesq approximation, one usually replace 1) the continuity equation by the assumption of a divergenceless velocity field; 2) in-situ density by a constant reference density \( \rho_0 \) everywhere in the equations of motion except when it appears in front of gravity. In principle, the validity of the Boussinesq approximation relies on the assumption that the density fluctuations are dominated by thermal (and haline in the oceanic case) effects rather than pressure effects, i.e., the fluid is assumed to be incompressible. In practice, however, this is not the case of oceanic motions, and most OGCMs use a nonlinear equation of state of the form \( \rho = \rho(S, \theta, p_r) \), where \( p_r \) is either a reference pressure depending upon depth only, or the full hydrostatic pressure, as recommended by Dewar et al. (1998). As a result of the approximations made, the hydrostatic Boussinesq primitive equations do not conserve mass, but volume instead. The energetics issue is more subtle, and is the topic of the present paper. In general, it seems well accepted that a closed energy budget only exists in special circumstances, and usually requires a modification of the thermodynamic properties of the fluid; Bannon (1996) expresses this result by the maxim: ‘the thermodynamics is slave to the dynamics’. Such a maxim is not recognized by current OGCMs, however, which in general retain the full nonlinear equation of state of seawater, and as a result do not conserve energy. In fact, it even turns out, as the present paper shows, that the hydrostatic Boussinesq equations in use in present OGCMs possess a number of spurious sources and sinks of energy that arise not only from the Boussinesq approximation itself, but also from other assumptions regarding the way turbulent mixing and diffusion are parameterized. As far as we know, a detailed knowledge of these spurious sources and sinks of energy is lacking, so that it will the main purpose of this paper to establish an appropriate framework to identify and quantify them.

An important motivation for the present work stems from the desire to find and develop objective norms to assess and measure the physical realism of atmospheric and oceanic models that are at the core of current climate models used for climate change prediction. So far, it seems fair to say that observations have been the primary criterion by which the performances of models have been measured, especially in the case of ocean models. Although it is widely recognized that such models possess numerous deficiencies, and that their parameterizations often lack a sound physical basis, the fact that the tunable parameters can be tuned to do a reasonable job at reproducing observations is often regarded as success by most ocean modellers; furthermore, an often heard opinion is that the main impediment to improving the ‘realism’ of ocean models comes from the difficulties of constraining the surface fluxes and initial distributions of temperature and salinity, and that the shortcomings in the physics of present models is probably of secondary importance. The problem, of course, is that the agreement with observations may be incidental and partly due to compensating errors, so that it is difficult to ascertain that current OGCMs have any predictive skills for a transient climate change in which errors may behave quite differently, and hence become more serious. Arguably, the only way to ascertain that the OGCMs will behave reasonably well in a different climate is by convincing ourselves that they are consistent with the fundamental laws and principles of physics, or if not, that the errors made are inconsequential.

At present, it is clear that most of existing OGCMs suffer from a number of undesirable physical shortcomings that often complicate the physical interpretation of their results. Some of the main debated issues concern: 1) the lack of local mass conservation that raises questions as to the accuracy of regional predictions for sea level change, e.g., Gregory (1993), Greathbach (1994), Lu (2001); 2) the use of a divergence-free velocity field that may be a possible significant source of spurious diapycnal mixing,
e.g., McDougall et al. (2002); 3) the lack of a well-defined energetic budget that make it difficult to use OGCMs to gain insight into the driving mechanisms of the actual oceans, e.g., Wunsch and Ferrari (2003), Huang (1998, 2004); 4) the use of a virtual salt flux instead of the natural boundary condition proposed by Huang (1993) which introduces an unphysical salt transport (see also Beron-Vera et al. (1999)). Obviously, all these shortcomings need to be critically assessed to better appreciate the kind of modifications possibly needed to improve the accuracy and physical realism of present OGCMs.

To the extent of our knowledge, it seems that the standard practice in establishing the property of energy conservation for approximate systems of equations is to do so for closed adiabatic systems, i.e., in absence of forcing and diabatic effects, e.g., Lilly (1996), Bannon (1996), Ingersoll (2005). Such an approach is therefore unsatisfactory if one is interested in identifying the possible spurious sinks and sources of energy in OGCMs, which are strongly diabatic systems exchanging heat and momentum with the atmosphere. In that case, what is needed is a way to systematically distinguish between conservative and nonconservative terms in the total energy equation, which is the main novelty of our approach. Here, we refer to a conservative term as one that can be expressed as the divergence of a mass or diffusive flux, so that its volume integral can be directly related to the mass and/or diffusive flux of the property through the surface enclosing the domain considered. In contrast, a nonconservative term appears as an internal destruction/production term in the equation considered. Mass, salinity, and total energy on the one hand, and entropy on the other hand, are probably the most well known examples of conservative and nonconservative quantities in oceanography. The determination of nonconservative effects in terms of the fluxes of the variable determining the thermodynamic state of the system is central to non-equilibrium thermodynamics, and many of the ideas in this paper borrow from the treatment of de Groot and Mazur (1962). Recently, McDougall (2003) discussed the issue of ‘conservativeness’ in relation with the standard practice in ocean modelling of treating potential temperature as a conservative quantity in spite of the second law of thermodynamics dictating that its evolution equation should possess a nonconservative term. McDougall (2003) argues that this can lead to errors in temperature of up to more than $1^\circ C$ in certain regions, but that this can be remedied by using a quantity called ‘conservative’ temperature instead, for which the maximum nonconservative terms are shown to be about two orders of magnitude smaller than for potential temperature. In this paper, we also find consequences on the energetics, by showing that treating potential temperature as a conservative quantity introduces errors in the total energy budget, but that these can be reduced by about one order of magnitude by using conservative temperature instead, which further supports the conclusions of McDougall (2003).

This paper is organized as follows. Section 2 first aims at further clarifying ideas about the distinction between conservative and non-conservative terms. Section 3 then goes on to apply these ideas to the case of the hydrostatic anelastic system of equations (Ogura and Phillips, 1962) of which the Boussinesq approximation stands as a particular case (Lilly, 1996; Bannon, 1996). This makes it possible to determine whether the anelastic approximation would allow for a reduction in the spurious sinks and sources of energy identified. As it turns out, this is not really the case. An important point is that the velocity field appearing in the equations is interpreted as the standard Eulerian velocity, in contrast with recent studies suggesting that it should be interpreted as a density-weighted velocity, e.g., Lu (2001) and McDougall et al. (2002). The consequences of this interpretation for the energetics are still unclear, and are left for future study. Finally, section 4 provides a discussion.
2. Energy and entropy balances in the non-Boussinesq equations

(a) Conservative and nonconservative equations

In this paper, we make extensive use of the concepts of conservative and nonconservative equations, with all examples discussed taking the following generic form:

\[
\frac{\partial (\rho Q)}{\partial t} + \nabla \cdot [\rho (Q + Q') v - F_q] = \rho \dot{Q}_{irr},
\]

where \( Q \) is an arbitrary scalar field defined per unit mass, \( \dot{Q}_{irr} \) is an internal production/destruction term, and \( F_q \) is a diffusive flux. The integral form of (1) is

\[
\frac{\partial}{\partial t} \int_D \rho Q dV = - \int_{\partial D} \rho (Q + Q') v \cdot n dS + \int_{\partial D} \rho F_q \cdot n dS + \int_D \rho \dot{Q}_{irr} dV
\]

where \( D \) is the domain considered, and \( \partial D \) is the boundary enclosing \( D \). Physically, the variations of \( Q \) with time are controlled by: 1) the mass flux of the field \( Q \) (with possibly the additional flux of \( Q' \)) through the control volume; 2) diffusive processes occurring through molecular exchanges without mass transport; 3) internal production/destruction owing to internal sources or sinks of \( Q \). Eq. (1) is referred to as conservative when \( \dot{Q}_{irr} = 0 \), and non-conservative otherwise. This definition is certainly reasonable in the case \( Q' = 0 \), while it may be debatable when \( Q' = 0 \). This definition is nevertheless adopted here for simplicity. The two main instances of conservative equations discussed in this paper are: 1) mass conservation (or continuity) equation for which \( Q = 1 \), \( Q' = 0 \), \( F_q = 0 \), and \( \dot{Q}_{irr} = 0 \); 2) total energy conservation equation for which \( Q = E \), \( Q' = p \), and \( \dot{Q}_{irr} = 0 \), where \( E \) is the total energy, and \( Q + Q' = B \) the so-called Bernoulli function. The main instance of a nonconservative equation is the evolution equation for entropy for which \( Q = \eta \), \( Q' = 0 \), while \( \dot{\eta}_{irr} \geq 0 \) by the second law of thermodynamics.

(b) Quantifying the nonconservative terms of nonconservative equations

To proceed, our starting point is the equations of motion for a rotating stratified ocean written under the form:

\[
\frac{\partial \mathbf{v}}{\partial t} + (2\Omega + \text{curl} \mathbf{v}) \times \mathbf{v} + \nabla \left( \frac{v^2}{2} \right) + \frac{1}{\rho} \nabla p = -\nabla \Phi + F
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

\[
\frac{D\eta}{Dt} = \frac{1}{\rho} \nabla \cdot (\rho F_{\eta}) + \dot{\eta}_{irr}
\]

\[
\frac{DS}{Dt} = \frac{1}{\rho} \nabla \cdot (\rho F_S)
\]

\[
\rho = \rho(S, \eta, P)
\]

where \( \mathbf{v} = (u, v, w) \) is the three-dimensional velocity field, \( p \) is pressure, \( \rho \) is in-situ density, \( \Omega \) is Earth’s rotation vector, \( \Phi = gz \) is the geopotential, with \( g \) the gravitational acceleration, and \( z \) the vertical coordinate pointing upward, \( F \) represents the effects of friction, \( F_{\eta} \) represents the diffusive flux of entropy, \( \dot{\eta}_{irr} \) is the irreversible production of entropy, and \( F_S \) is the diffusive flux of salt.
By multiplying the momentum equations (3) by $\rho v$, and the mass conservation equation (4) by $gz$, one obtains after some standard manipulation the following evolution equations for kinetic and gravitational potential energy respectively:

$$\frac{\partial}{\partial t} \left( \rho \frac{v^2}{2} \right) + \nabla \cdot \left[ \rho \left( \frac{v^2}{2} + \frac{P}{\rho} \right) v - \rho F_{ke} \right] = -\rho gw + P \nabla \cdot v - \rho \varepsilon \tag{8}$$

$$\frac{\partial (\rho gz)}{\partial t} + \nabla \cdot (\rho gz v) = \rho gw \tag{9}$$

where we wrote $\rho v \cdot F = \nabla \cdot (\rho F_{ke}) - \rho \varepsilon$ as the sum of a conservative and nonconservative term, with $F_{ke}$ a diffusive flux of kinetic energy, and $\rho \varepsilon$ the kinetic energy dissipation rate, i.e., the work against friction. Summing (8) and (9) gives the evolution equation for the mechanical energy of the system:

$$\frac{\partial E_{mec}}{\partial t} + \nabla \cdot F_{mec} = P \nabla \cdot v - \rho \varepsilon = -\frac{P}{\rho} \frac{D \rho}{Dt} - \rho \varepsilon, \tag{10}$$

where the latter equality makes use of the continuity equation (4). By construction, the terms on the right-hand side must be the local conversion terms with internal energy, and physically represent the work of contraction/expansion of fluid parcels, and the conversion into heat of dissipated kinetic energy.

One standard differential expression for internal energy is the following:

$$dU = \mu dS + T d\eta + \frac{P}{\rho^2} dp \tag{11}$$

where $\mu$ is the relative chemical potential of seawater, e.g., Feistel and Hagen (1995). Using Eqs. (6), (5), and (4), leads to:

$$\frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U v) = \mu \nabla \cdot (\rho F_S) + T (\nabla \cdot (\rho F_\eta) + \rho \dot{\eta}_{irr}) - P \nabla \cdot v \tag{12}$$

which can be rewritten as follows:

$$\frac{\partial (\rho U)}{\partial t} + \nabla \cdot [\rho (U v - \mu F_S - T F_\eta)] = -\rho [F_S \cdot \nabla \mu + F_\eta \cdot \nabla T - T \dot{\eta}_{irr}] - P \nabla \cdot v. \tag{12}$$

Summing (8), (9), and (12) leads to:

$$\frac{\partial E}{\partial t} + \nabla \cdot F_E = \dot{E}_{irr}, \tag{13}$$

where the total energy $E$, its flux $F_E$, and the sink/source term $\dot{E}_{irr}$ are defined by

$$E = \rho \left( \frac{v^2}{2} + gz + U \right), \tag{14}$$

$$F_E = \rho \left[ \left( \frac{v^2}{2} + gz + U + \frac{P}{\rho} \right) v - \rho F_{ke} - \mu F_S - T F_\eta \right], \tag{15}$$

$$\dot{E}_{irr} = \rho [T \dot{\eta}_{irr} - F_S \cdot \nabla \mu - F_\eta \cdot \nabla T - \varepsilon]. \tag{16}$$

Here one sees immediately that in order to be consistent with the law of total energy conservation, the nonconservative term $\dot{E}_{irr}$ appearing in Eq. (13) must vanish. If we assume that the expressions for the salinity and entropy fluxes are given, this necessarily
requires the irreversible production of entropy to be
\[ \dot{\eta}_{\text{irr}} = \frac{\mathbf{F}_S \cdot \nabla \mu + \mathbf{F}_\eta \cdot \nabla T + \varepsilon}{T}. \] (17)

By the second law of thermodynamics, we know that \( \dot{\eta}_{\text{irr}} \) must be strictly positive outside equilibrium conditions; this is well-known to imply restrictions on the allowable forms of the salinity and entropy fluxes, e.g., see de Groot and Mazur (1962), Salmon (1986), Gregg (1984), Davis (1994).

In the oceanic literature, nonconservative effects are often estimated by considering the mixing of two samples of seawater with differing T/S properties at constant pressure, e.g., see Fofonoff (1962), Salmon (1986), McDougall (2003). The above derivation generalizes the idea to the case of the continuous system, and for non-resting states. One difference between Eq. (17) and those published in the literature is that the latter uses enthalpy fluxes rather than entropy fluxes. To recover the classical expression, simply use the total differential for enthalpy:

\[ dH = dS + T \, d\eta + \frac{dP}{\rho}, \] (18)

from which we see that the enthalpy flux must be related to the entropy flux by:

\[ \mathbf{F}_h = \mu \mathbf{F}_S + T \mathbf{F}_\eta, \] (19)

which allows one to rewrite the entropy production term as follows:

\[ \dot{\eta}_{\text{irr}} = \mathbf{F}_S \cdot \nabla \left( \frac{\mu}{T} \right) - \mathbf{F}_h \cdot \nabla \left( \frac{1}{T} \right) + \frac{\varepsilon}{T}. \] (20)

which the reader may find more familiar, e.g., Gregg (1984).

(c) Irreversible production of potential and conservative temperature

We now apply the above method to the determination of the nonconservative terms in the evolution equations for potential temperature \( \theta \) and conservative temperature \( \Theta \), which are usually preferred to entropy in ocean general circulation models. To proceed, let us replace the entropy equation (5) by an equivalent one in terms of either \( \theta \) or \( \Theta \). The method being as above, we leave the technical details to Appendix B, and simply list the results below. The nonconservative properties of potential and conservative temperature were previously quantified by McDougall (2003) by considering the mixing of two parcels of seawater with different T/S characteristics at constant pressure, along the lines pioneered by Fofonoff (1962). The following approach is somewhat more general.

(i) Irreversible production of potential temperature

One possible way to define potential temperature is in terms of entropy as follows:

\[ \eta(S, T, p) = \eta(S, \theta(S, T, p, p_r), p_r) = \eta_r(S, \theta), \] (21)

e.g., Feistel (2003), where \( p_r \) is a reference pressure usually taken at the surface. In that case, the results of Appendix B show that we have:

\[ \dot{\theta}_{\text{irr}} = \frac{\theta}{TC_{\text{pr}}} \left[ \mathbf{F}_S \cdot \nabla \left( \mu - T \frac{\partial \mu_r}{\partial \theta} \right) + \mathbf{F}_\theta \cdot \nabla \left( \frac{TC_{\text{pr}}}{\theta} \right) + \varepsilon \right]. \] (22)

Interestingly, it does not appear possible to ascertain that \( \dot{\theta}_{\text{irr}} \) should be sign definite, unlike for entropy.
(ii) **Irreversible production of conservative temperature**  
Conservative temperature is defined by  
\[ \Theta = H / C_{\rho 0}, \]
where  \( H \) is potential enthalpy, and  \( C_{\rho 0} \) a constant value of the specific heat capacity, as discussed by McDougall (2003); a definition for  \( H \) is  
\[ H(S, T, p, p_r) = H(S, \theta(S, T, p, p_r), p_r), \]
(Feistel, 2003). In that case, Appendix B shows that we have:  
\[ \dot{\Theta}_{irr} = \frac{1}{C_{\rho 0}} \left[ \mathbf{F}_S \cdot \nabla \left( \mu - \frac{T \mu_{irr}}{\Theta} \right) + \mathbf{F}_\Theta \cdot \nabla \left( \frac{TC_{\rho 0}}{\Theta} \right) \varepsilon \right]. \]
(24)

In comparison with  \( \dot{\Theta}_{irr} \),  \( \dot{\Theta}_{irr} \) vanishes at the surface, and will in general increase with pressure. McDougall (2003) shows that the maximum error made in assuming  \( \dot{\Theta}_{irr} = 0 \) is at least one hundred times smaller than the maximum error made when assuming  \( \dot{\Theta}_{irr} = 0 \). On a pointwise basis, however,  \( \dot{\Theta}_{irr} \) will not necessarily be much smaller than  \( \dot{\Theta}_{irr} \), and can even be greater sometimes; this has to be kept in mind to properly interpret some of the results obtained in the following section.

### 3. APPLICATION TO BOUSSINESQ-EQUATIONS MODELS

#### (a) Hydrostatic models using the anelastic approximation

Being now equipped with the necessary tools to estimate the non conservative terms arising in the thermodynamic equations of oceanic motion, we can turn to the problem of identifying and quantifying the spurious sources and sinks of energy in the hydrostatic Boussinesq equations in use in many current  \( z \)-coordinate models. In order to be as general as possible, and examine some of the possible ways to reduce the magnitude of existing errors, the following discussion considers the weak form of the Boussinesq approximation known as the anelastic approximation, which contains the standard (or strong) form of the Boussinesq approximation as a special case. Specifically, the equations of motion considered are as follows:

\[ \frac{\partial \mathbf{u}}{\partial t} + (f + \zeta) \mathbf{k} \times \mathbf{u} + w \frac{\partial \mathbf{u}}{\partial z} + \nabla_H \left( \frac{\mathbf{u}^2}{2} \right) + \frac{1}{\rho_0} \nabla_H P = \mathbf{F}_H, \]
(25)

\[ \frac{\partial P}{\partial z} = -\rho g, \]
(26)

\[ \nabla \cdot (\rho_0 \mathbf{v}) = 0, \]
(27)

\[ \frac{D\theta}{Dt} = \frac{1}{\rho_0} \nabla (\rho_0 \cdot \mathbf{F}_\theta) + \dot{\theta}_{irr}, \]
(28)

\[ \frac{DS}{Dt} = \frac{1}{\rho_0} \nabla \cdot (\rho_0 \mathbf{F}_S), \]
(29)

\[ \rho = \rho(S, \theta, P_0(z)), \]
(30)

where  \( \nabla_H \) is the horizontal  \( \nabla \) operator,  \( \zeta \) is the vertical component of the relative vorticity,  \( f = 2\Omega \sin \phi \) is the Coriolis parameter, with  \( 2\Omega \) twice Earth’s rotation rate, and  \( \phi \) latitude. The temperature-like variable in (28) refers to either potential or conservative temperature; a nonconservative term has been included for tracking purposes, but it is usually neglected in numerical models. The functions  \( \rho_0 \) and  \( P_0 \) are a reference density...
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and hydrostatic pressure profiles, depending upon depth only, and determined from:

$$\frac{\partial P_0}{\partial z} = -g\rho_0, \quad \frac{1}{\rho_0} \frac{d\rho_0}{dz} = -g c_0^2 = -\frac{1}{H_{0s}}$$  \hspace{1cm} (31)

where $c_0^2(z)$ is a reference profile for the speed of sound squared, so that $H_{0s}$ has the

dimension of a depth. Since $c_0 = O(1500 \text{ m/s})$, we have $H_{0s} = O(225 \text{ km})$ which is
everywhere very much larger than the total ocean depth. This implies that $\rho_0$ and $P_0$

are very nearly a linear and quadratic function of depth respectively. In the anelastic

approximation, the divergence of $\mathbf{v}$ is given by:

$$\nabla \cdot \mathbf{v} = -\frac{w}{\rho_0} \frac{d\rho_0}{dz} = \frac{w}{H_{0s}}.$$  \hspace{1cm} (32)

The strong form of the Boussinesq approximation, for which $\rho_0 = \text{cst}$ and $\nabla \cdot \mathbf{v} = 0$,
corresponds to the limit $H_{0s} = +\infty$. Note that the pressure appearing in the equation

of state is assumed to depend on depth only, following widespread practice. We ac-

knowledge the result by Dewar et al. (1998) that this may introduce serious errors in

the geostrophic velocity, but we also find that the remedy proposed, i.e., to replace

$P_0(z)$ by the full hydrostatic pressure, while suppressing the abovementioned errors,

also introduces additional spurious sources/sinks of energy in the total energy equation,
cancelling some of the beneficial effects of the approach, a point overlooked so far.

The relevant evolution equations for kinetic energy, gravitational potential energy,

and total mechanical energy in the context of the present approximations are respectively
given by:

$$\frac{\partial}{\partial t} \left( \frac{\rho_0 \mathbf{u}^2}{2} \right) + \nabla \cdot \left[ \rho_0 \left( \frac{\mathbf{u}^2}{2} + \frac{P}{\rho_0} \right) \mathbf{v} - \rho_0 \mathbf{F}_{ke} \right] = -\rho g w + P \nabla \cdot \mathbf{v} - \rho_0 \varepsilon, \hspace{1cm} (33)$$

$$\frac{\partial (\rho g z)}{\partial t} + \nabla \cdot (\rho g z \mathbf{v}) = \rho g w + g z \hat{R}_{irr}, \hspace{1cm} (34)$$

$$\frac{\partial \mathcal{E}_{mec}}{\partial t} + \nabla \cdot \mathbf{F}_{mec} = P \nabla \cdot \mathbf{v} + g z \hat{R}_{irr} - \rho_0 \varepsilon,$$  \hspace{1cm} (35)

where the work of viscous processes has been decomposed as the sum of a conservative

and nonconservative term as: $\rho_0 \mathbf{u} \cdot \mathbf{F}_H = \nabla \cdot (\rho_0 \mathbf{F}_{ke}) - \rho_0 \varepsilon$, while $\hat{R}_{irr}$ is the spurious

source/sink of mass arising from the Boussinesq approximation, and defined by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \hat{R}_{irr}.$$  \hspace{1cm} (36)

By inspecting (33) and (34), it appears the weak and strong forms of the Boussinesq
approximation greatly alters some of the original energy conversion terms, specifically:

1. The conversion term between kinetic to internal energy that is due to the work

of contraction or expansion of the fluid parcels against the pressure field has become inaccurate in (33), for it vanishes identically in the strong Boussinesq

approximation, while involving only the compression/expansion work against the

background state in the anelastic approximation;

2. The work of expansion/contraction that disappears from the kinetic energy equation (33) actually reappears in the gravitational potential energy equation (34) in

a somewhat distorted form, with the term $P/\rho_0$ being replaced by $g z$, and the

expansion/contraction appearing as the mass residual $\hat{R}_{irr}$. 
These considerations indicate that although the original Boussinesq approximation was
intended to decouple the evolution of mechanical and internal energies, e.g., Mihaljan
(1962), a coupling term between mechanical and internal energy can be constructed,
i.e., the term $C$ in (35). This conversion term $C$ takes the following forms depending on
whether the weak ($C = C_{\text{anelastic}}$) or strong ($C = C_{\text{boussinesq}}$) form of the Boussinesq
approximation are considered:

$$
C_{\text{boussinesq}} = g z \frac{D \rho}{Dt} = -\frac{P_{\text{lin}}}{\rho_0} \frac{D \rho}{Dt}, \quad (37)
$$

$$
C_{\text{anelastic}} = -\frac{P_{\text{lin}}}{\rho_0} \frac{D \rho}{Dt} + (P + \rho g z) \nabla \cdot \mathbf{v} = -\frac{P_{\text{lin}}}{\rho_0} \frac{D \rho}{Dt} - \frac{(P + \rho g z)}{\rho_0} \frac{D \rho}{Dt}. \quad (38)
$$

Since $C_{\text{boussinesq}}$ and $C_{\text{anelastic}}$ are visibly of the same order of magnitude, and in
both cases close to the actual energy conversion term between mechanical and internal
energy, it follows that closing the energy budget for the primitive Boussinesq equations
actually requires adding in an evolution equation for internal energy. However, because
neither $C_{\text{boussinesq}}$ nor $C_{\text{anelastic}}$ are exactly identical to the actual conversion term
$P D \rho / Dt$, the total energy budget cannot be exactly closed. In other words, there
will always be a discrepancy between the amount of internal energy expected by the
mechanical energy part of the Boussinesq equations and that occurring as the result of
the actual evolution of temperature and salinity. The following seeks to estimate the
magnitude of this discrepancy.

An evolution equation for the total energy is sought by distinguishing between the
cases where potential or conservative temperature is the relevant temperature variable.
From the results of Appendix B, the evolution equation for internal energy can be
respectively written in terms of potential and conservative temperature as follows:

$$
\frac{DU}{Dt} = \begin{cases} 
\theta : \left( \mu - T \frac{\partial \mu}{\partial \theta} \right) \frac{DS}{Dt} + \frac{TC_{pr}}{\theta} \frac{D \theta}{Dt} + \frac{P}{\rho^2} \frac{D \rho}{Dt}, \\
\Theta : \left( \mu - T \frac{\partial \mu}{\partial \theta} \right) \frac{DS}{Dt} + \frac{TC_{pr}}{\theta} \frac{D \theta}{Dt} + \frac{P}{\rho^2} \frac{D \rho}{Dt}
\end{cases}, \quad (39)
$$

so that the sought-for evolution equation for the total energy $E$ takes the form (cf. Eqs.
(13)- (16)):

$$
\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F}_E = S - \rho_0 \varepsilon + \left( \rho g z + \frac{\rho_0 P}{\rho} \right) \frac{1}{\rho} \frac{D \rho}{Dt} + (P + \rho g z) \nabla \cdot \mathbf{v}. \quad (40)
$$

The considerations developed above make it clear that the right-hand side of (40) repre-
sents the spurious sources and sinks of energy arising from the various approximations
made in the present context. Here, it is expressed as the sum of three terms $S$, $\rho_0 \varepsilon$ and
$C_2$, with the meaning of $S$ and $C_2$ detailed below.

The term $S$ arises as the result of neglecting the non-conservative production/destruction
of potential/conservative temperature; its precise form depends on which temperature
variable is used, with

$$
S = \begin{cases} 
\theta : -\rho_0 \left[ \mathbf{F}_S \cdot \nabla \left( \mu - T \frac{\partial \mu}{\partial \theta} \right) + \mathbf{F}_\theta \cdot \nabla \left( \frac{TC_{pr}}{\theta} \right) \right] + \rho_0 \frac{TC_{pr}}{\theta} \Theta_{irr}, \\
\Theta : -\rho_0 \left[ \mathbf{F}_S \cdot \nabla \left( \mu - T \frac{\partial \mu}{\partial \theta} \right) + \mathbf{F}_\theta \cdot \nabla \left( \frac{TC_{pr}}{\theta} \right) \right] + \rho_0 \frac{TC_{pr}}{\theta} \Theta_{irr}
\end{cases} \quad (41)
$$
In reality, one would fix the value of the non-conservative terms \( \dot{\theta}_{\text{irr}} \) and \( \dot{\Theta}_{\text{irr}} \) so that the sum \( S - \rho_0 \varepsilon \) vanishes by virtue of the arguments developed in Section 2. Since \( \dot{\theta}_{\text{irr}} \) and \( \dot{\Theta}_{\text{irr}} \) are taken to vanish in OGCMs, it follows that the sum \( S - \rho_0 \varepsilon \) will differ from zero in general, and hence contributes to the spurious generation/destruction of energy in OGCMs.

The term \( C_2 \) represents the discrepancy between the work of expansion/contraction that actually interact with the mechanical energy of Boussinesq models, and the expansion/contraction determined by the actual evolution of the temperature and salinity fields. The form of \( C_2 \) varies with the form (weak or strong) of the Boussinesq approximation; furthermore, \( C_2 \) can be regarded to be made of a diabatic and adiabatic part, i.e., we can write

\[
C_2 = C_{\text{diabatic}} + C_{\text{adiabatic}}. \tag{42}
\]

Using the fact that the equation of state \( \rho = \rho(S, \theta, p_0(z)) \) implies,

\[
\frac{D\rho}{Dt} = \rho_S \frac{DS}{Dt} + \rho_\theta \frac{D\theta}{Dt} - \frac{\rho_0 g}{c_s^2} w, \tag{43}
\]

we have

\[
C_{\text{diabatic}} = \left( gz + \frac{\rho_0 P_0}{\rho^2} \right) \left( \frac{\rho_\theta}{\rho_0} \nabla : (\rho_0 \mathbf{F}_S) + \frac{\rho_\theta}{\rho_0} \nabla : (\rho_0 \mathbf{F}_\theta) + \rho_0 \dot{\theta}_{\text{irr}} \right) \tag{44}
\]

for potential temperature, with a similar expression for conservative temperature. Note that \( C_{\text{diabatic}} \) is independent of the form (weak or strong) of the Boussinesq approximation made, unlike \( C_{\text{adiabatic}} \), for which we have:

\[
C_{\text{adiabatic}} = \begin{cases} 
\text{weak :} & \frac{w}{H_{\infty}} \left[ P \left(1 - \frac{\rho_0 H_{\infty}}{p_0 H_{\infty}}\right) + \rho g z \left(1 - \frac{\rho_0 H_{\infty}}{p_0 H_{\infty}}\right) \right] \\
\text{strong :} & \frac{w}{H_{\infty}} \left[ \rho_0 g z + \frac{\rho_0 P}{\rho^2} \right]
\end{cases} \tag{45}
\]

Note here that the expressions (45) would be significantly more complicated had we assumed the equation of state to depend on the actual hydrostatic pressure, rather than upon the reference pressure \( p_0(z) \), as recommended by Dewar et al. (1999). While it may be true that using \( p_0(z) \) rather than \( p \) incurs non-negligible errors in the velocity field, as argued by Dewar et al. (1999), the present considerations suggest that this will come at the price of introducing additional spurious sources/sinks of energy in the total energy budget, so that the actual benefit of using the full hydrostatic pressure is unclear.

The energy balance equation (40), along with the expressions (41), (44), and (45), are a central basis of this paper, as they form the basis for identifying and possibly improving the energy budget of current ocean general circulation models. These expressions show that the energy budget of the Boussinesq equations is not closed because of:

1. Treating potential or conservative temperature as conservative quantities (i.e., assuming \( \dot{\theta}_{\text{irr}} = \dot{\Theta}_{\text{irr}} = 0 \)), which results in the non-vanishing of \( S - \rho_0 \varepsilon \).
2. The interaction of the Boussinesq approximation with the turbulent mixing parameterizations of temperature and salinity, as measured by \( C_{\text{diabatic}} \); the magnitude of this term is the same in both the Boussinesq and anelastic approximations, and probably depends little on the particular choice of the temperature variable;
3. The interaction of the Boussinesq approximation with the retention of compressibility effects in the equation of state, as measured by \( C_{\text{adiabatic}} \); this term is the only one significantly reduced by the anelastic approximation;
4. Additionally, other errors are expected to arise in numerical implementations of the Boussinesq equations, as the result of the time-stepping method used for instance. A synthesis of the energetics for the Boussinesq approximation is given in Fig. 1. The following paragraph attempts a quantification of the various spurious sources/sinks of energy in (40).
The decomposition of the terms $S_\theta$ and $S_\Theta$

<table>
<thead>
<tr>
<th>Term</th>
<th>Hor. part (GW)</th>
<th>Vert. part (GW)</th>
<th>Total (GW)</th>
<th>Vol. aver. ($Wm^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_\theta$</td>
<td>3.5</td>
<td>$-8.25$</td>
<td>$-4.5$</td>
<td>$-3.10^{-9}$</td>
</tr>
<tr>
<td>$S_\Theta$</td>
<td>$-0.35$</td>
<td>0.75</td>
<td>0.2</td>
<td>$1.7 \times 10^{-10}$</td>
</tr>
</tbody>
</table>

Estimates based on the values $K_H = 2.10^3 m^2s^{-1}$ and $K_v = 10^{-4} m^2s^{-1}$. The volume averaged values are computed using $V_{ocean} = 1,370$ million $km^3$.

(b) Order of magnitude of the nonservative terms in (40)

(i) The term $S$ In order to estimate this term, we assume that the salt and heat fluxes take the following form:

$$F_Q = K_H \nabla_h Q + K_v \frac{\partial Q}{\partial z}$$

where $Q$ is either $S$, $\theta$, or $\Theta$, with $K_H$ and $K_v$ horizontal and vertical turbulent diffusivities respectively. The specific form of $S$ depends on whether potential or conservative temperature is used, with:

$$\frac{S_\theta}{\rho_0} = -K_H \left[ \nabla_h S \cdot \nabla \mu^* + \nabla_h \theta \cdot \nabla \left( \frac{TC_{pr}}{\theta} \right) \right] - K_v \left[ \frac{\partial S}{\partial z} \frac{\partial \mu^*}{\partial z} + \frac{\partial \theta}{\partial z} \frac{\partial \left( \frac{TC_{pr}}{\theta} \right)}{\partial z} \right]$$

(46)

for potential temperature, and

$$\frac{S_\Theta}{\rho_0} = -K_H \left[ \nabla_h S \cdot \nabla \mu^{**} + \nabla_h \Theta \cdot \nabla \left( \frac{TC_{\theta\theta}}{\theta} \right) \right] - K_v \left[ \frac{\partial S}{\partial z} \frac{\partial \mu^{**}}{\partial z} + \frac{\partial \Theta}{\partial z} \frac{\partial \left( \frac{TC_{\theta\theta}}{\theta} \right)}{\partial z} \right]$$

(47)

for conservative temperature. To simplify the above expressions, we defined $\mu^* = \mu - T \partial \theta / \mu_r$, and $\mu^{**} = \mu - T \mu_r / \theta$. The two terms $S_\theta$ and $S_\Theta$ were estimated for the world ocean by using the climatological values for temperature and salinity provided by the Levitus database. Table 1 summarizes the result for the particular values $K_H = 2000 m^2s^{-1}$ and $K_v = 10^{-4} m^2s^{-1}$, by separating the horizontal and vertical parts respectively proportional to $K_H$ and $K_v$, the volume integrated value of the sum in GigaWatts, and the volume averaged value assuming an oceanic volume of $V_{ocean} = 1,370$ million $km^3$.

With the above values of $K_H$ and $K_v$, $S_\theta$ and $S_\Theta$ appear respectively as a sink and source of energy, although it is difficult to say whether this should always hold. The situation would be reverted, however, by choosing the more realistic value of $K_v = 10^{-5} m^2s^{-1}$ for the vertical/diapycnal diffusivity. Such a behaviour could perhaps contribute to make the model numerically unstable.

(ii) “Dissipation” of kinetic energy The dissipation of kinetic energy, or more exactly the transfer of large-scale energy to the unresolved scales, is assumed to take the form:

$$\rho_0 \varepsilon = \rho_0 A_H (\| \nabla_h u \|^2 + \| \nabla_h v \|^2) + \rho_0 A_V (u_z^2 + v_z^2).$$

In a coarse resolution model, $A_H$ may vary between about 2000 m$^2$/s near the equator, to 40000 m$^2$/s at high-latitudes. Using a shear of 1 m/s over 10$^7$ m, and the highest end value for $A_H$ yields

$$\rho_0 \varepsilon = O(4.10^{-7}) Wm^{-3} > 0,$$

which is always positive, and hence represent a sink of kinetic energy which most likely contributes to the numerical stability of the model. Multiplying this estimate by the
ocean volume to get an idea of the volume-integrated value would yield \( V_{ocean} \rho_0 \epsilon = O(5.2 \times 10^{11} \text{W}) = 0.5 \text{TW} \), which is close to the value given in Wunsch and Ferrari (2003) for the kinetic energy dissipation rate below the mixed layer.

**(iii) Terms arising from the Boussinesq approximation: \( C_{\text{diabatic}} \) and \( C_{\text{adiabatic}} \).** In order to estimate these terms, we use the result that the hydrostatic pressure varies linearly at leading order, i.e., there exists some reference density \( \rho^* \) so that \( P = \rho^* g z \). As a result, we can write:

\[
gz + \frac{\rho_0 P}{\rho^2} \approx gz \left( 1 - \frac{\rho_0 \rho^*}{\rho^2} \right) = O \left( \frac{gz \delta \rho}{\rho} \right),
\]

for some \( \delta \rho \) representing the amount by which \( \rho \) differs from \( (\rho^* \rho_0)^{1/2} \) over depth. To estimate \( C_{\text{diabatic}} \), we assume the divergence terms to be dominated by the temperature contribution. The heat fluxes at the surface are typically of the order \( 10^5 - 50 \text{ W.m}^{-2} \), or in thermal units \( 2.5 - 12.5 \times 10^{-6} \text{ K.m.s}^{-1} \). Assuming that the heat flux is absorbed within the first 1000 m, and using a typical value of \( \rho_b = 0.2 \text{ kg.m}^{-3} \cdot \text{K}^{-1} \), we have:

\[
\frac{\rho_0}{\rho_0} \nabla \cdot (\rho_0 \mathbf{F}_B) = O \left( 0.5 - 2.5 \times 10^{-9} \text{ kg.m}^{-3} \cdot \text{s}^{-1} \right).
\]

Finally, choosing \( gz = 10^4 \text{ m}^2\text{s}^{-2} \), and \( \delta \rho / \rho = 10^{-2} \) yields

\[
C_{\text{diabatic}} = O \left( 0.5 - 2.5 \times 10^{-7} \text{ W.m}^{-3} \right),
\]

which is about the same order of magnitude as the kinetic energy dissipation term. In general, however, this term will not be sign definite. With regard to \( C_{\text{adiabatic}} \), one may show that it scales as

\[
C_{\text{adiabatic}} = O \left( \frac{\delta \rho gwz}{H_s} \right)
\]

where \( \delta \rho \) scales as above for the strong Boussinesq approximation, while it scales as the typical difference of \( \rho \) and \( \rho_0 \) in the anelastic approximation. One should therefore take somewhat smaller values for \( \delta \rho \) in the anelastic approximation than in the Boussinesq approximation. To represent the possible range, we use \( \delta \rho = 1 - 10 \text{ kg.m}^{-3} \), \( z = 1000 \text{ m} \), \( H_s = 2.25 \times 10^5 \text{ m} \), \( w = 10^{-7} \text{ m.s}^{-1} \) to arrive at:

\[
C_{\text{adiabatic}} = O \left( 0.5 - 5 \times 10^{-8} \text{ W.m}^{-3} \right),
\]

which appears comparable, although perhaps somewhat smaller \( C_{\text{diabatic}} \).

(c) **Interpretation**

The above estimates provide only rough values for the different sinks and sources of energy expected to occur in numerical implementations of the type of hydrostatic Boussinesq approximation currently used by most oceanic general circulation models. More detailed estimates can only be obtained by using the output of such a model, which is beyond the scope of this preliminary study. In order to get some sense of the importance of such values, it is interesting to compare them with the numbers entering the “Strawman energy budget” of Wunsch and Ferrari (2003) (see their Fig. 5). In this diagram, the authors attempted to list the main different energy reservoirs existing in the oceans, as well as the magnitude of the energy transfers between them. The numbers range from about 20 TW for the wind input to the large scale circulation to about 0.1 TW for the maintenance of the abyssal stratification by mixing from meso-scale
eddies for instance. It follows, therefore, that the spurious sinks and sources of energy appearing in the right-hand side of (40) are of the same order of magnitude as many of the physical processes entering the strawman energy budget of Wunsch and Ferrari (2003). It must therefore be concluded that it would not make much sense to attempt improving the physical realism of current ocean models without removing some of the spurious sources and sinks of current ocean models.

4. Discussion

This paper establishes that in order to correctly interpret the energetics of current OGCMs, one needs to be aware that the spurious sinks of energy that are known to exist in Boussinesq ocean models have at least three main causes: 1) treating potential and conservative temperature as conservative quantities, and not converting the kinetic energy dissipation into heat; 2) the interactions between the parameterized diabatic terms and the Boussinesq approximation; 3) the inconsistency stemming from assuming the fluid incompressible while at the same time retaining the pressure dependence in the equation of state of seawater. In addition, there will be other errors stemming from the particular numerical implementation of the primitive Boussinesq equations, in particular associated with the time-stepping method. The present analysis, however, does not permit to determine whether the numerical error is likely to be larger or smaller than the physical errors. Hopefully, the present framework can provide guidance to design suitable diagnostics to answer this question.

Apart from the kinetic energy dissipation term and perhaps $S_{\Delta}$, none of the other nonconservative energy terms appear to be sign definite, so that although they may be locally significant, they are likely to be less so when integrated over the whole domain owing to cancellation effects. For this reason, one may argue that the main problem with the energetics of current OGCMs concerns less the global balance than the local ones. Based on the present results, the physical realism of local energy balances in current Boussinesq OGCMs appears to be dubious on the basis that: 1) the work of expansion/contraction is at the “wrong” place, having been transferred from the kinetic energy equation to the gravitational potential energy equation; 2) the magnitude of some of the unphysical energy conversion terms identified in this paper appears to be comparable with some of the energy conversion terms associated with actual physical processes, based on the numbers published in Wunsch and Ferrari (2003) when converted into local estimates by dividing by the total ocean volume.

A definite understanding of the importance of correct local energy balances on the accuracy and quality of model simulations probably must await actually improving such balances and testing the impacts of such improvements in concrete cases. A question of importance is whether these errors can impact transient simulations of climate change, as well as transitions toward different circulation regimes which we expect to be characterized by a different repartitions among the different energy reservoirs. What kind of corrections need to be implemented exactly, however, is far from clear and requires further work. Recently, Lu (2001) and McDougall et al. (2002) proposed a way to improve local mass conservation in Boussinesq models aiming at minimally affecting existing code architectures, but they did not investigate the consequences on models’ energetics. Surprisingly, we find that using the anelastic approximation would probably little improve ocean energetics, contrary to our expectations, while probably requiring significant modifications to existing codes architecture. By contrast, using

† To compare these numbers with the local estimates, note that 1.37 TW yields a mean value of $10^{-6}$ W.m$^{-3}$ by dividing by the ocean volume $V_{\text{ocean}} = 1.37 \times 10^{18}$ m$^3$. 
conservative temperature instead of potential temperature as advocated by McDougall (2003) can reduce one of the spurious energy term by one order of magnitude while being straightforward to implement. Finally, one possibility suggested by several authors would be to rewrite the equations in pressure coordinates, e.g., de Szoeke and Samelson (2002). We hope that the present work, and the systematic framework it provides to identify nonconservative terms in the energy budget, will help stimulate further work in the topic, and contribute to improving the physical realism and consistency of ocean general circulation models.

ACKNOWLEDGEMENTS

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APPENDIX A

Thermodynamic coefficients

This section provides the relevant expressions for the haline contraction, thermal expansion, and compressibility coefficients corresponding respectively to the use of entropy, potential temperature, and conservative temperature in the equation of state, retaining in each case salinity and pressure as the two other independent thermodynamic coordinates. We start with entropy $\eta$, and thus write the total differential of $\rho = \rho(S, \eta, p)$ as follows:

$$dp = \rho_S dS + \rho_\eta d\eta + \frac{1}{c_s^2} dp,$$

where $c_s$ is the speed of sound. Entropy itself can be regarded as a function of state in salinity, in-situ temperature, and pressure, its total differential being:

$$d\eta = -\frac{\partial \mu}{\partial T} dS + \frac{C_p}{T} dT - \frac{\alpha}{\rho} dp,$$

e.g., Feistel and Hagen (1993), where $\alpha$ is the “standard” thermal expansion coefficient, and $\mu$ the chemical potential of seawater. By inserting (A.2) into (A.1), one obtains

$$dp = \left(\rho_S - \rho_\eta \frac{\partial \mu}{\partial T}\right) dS + \rho_\eta \frac{C_p}{T} dT + \left(\frac{1}{c_s^2} - \rho_\eta \frac{\alpha}{\rho}\right) dP,$$

which is the total differential of $\rho$ in the $(S, T, p)$ coordinate system. Proceeding by identification, as in Tailleux et al. (2005), one obtains:

$$\rho_\eta = -\frac{\alpha \rho T}{C_p},$$

$$\rho_S = \rho \beta - \frac{\alpha \rho T}{C_p} \frac{\partial \mu}{\partial T},$$
To establish an expression for the thermodynamic coefficients in the \((S, \theta, p)\) system, we use the following implicit definition of potential temperature:

\[
\eta(S, T, p) = \eta_r(S, \theta(S, T, p, p_r), p_r),
\]  

(A.7)
e.g., Feistel (2005), where \(p_r\) is a reference pressure, usually assumed at the surface. If we differentiate this expression assuming \(p_r\) fixed, we can write:

\[
d\eta = -\frac{\partial \mu_r}{\partial \theta} dS + \frac{C_{pr}}{\theta} d\theta.
\]  

(A.8)

Then, inserting (A.8) into (A.1) yields,

\[
d\rho = \left( \rho_S - \rho_\eta \frac{\partial \mu_r}{\partial \theta} \right) dS + \rho_\eta \frac{C_{pr}}{\theta} d\theta + \frac{1}{c_s^2} dp.
\]  

(A.9)

Using the previous results for \(\rho_S\) and \(\rho_\eta\) yields

\[
\beta_\theta = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial S} \right)_{\theta,p} = \beta + \frac{\alpha T}{C_p} \left( \frac{\partial \mu_r}{\partial \theta} - \frac{\partial \mu}{\partial T} \right) = \beta \left[ 1 + \frac{\alpha T}{\beta C_p} \left( \frac{\partial \mu_r}{\partial \theta} - \frac{\partial \mu}{\partial T} \right) \right]
\]  

(A.10)

\[
\alpha_\theta = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial \theta} \right)_{S,p} = \left( \frac{T C_{pr}}{\theta C_p} \right) \alpha
\]  

(A.11)

which represent the sought-for expressions for the haline contraction and thermal expansion coefficients in the \((S, \theta, p)\) space. To extend these results to the \((S, \Theta, p)\) space, where \(\Theta\) is the conservative temperature of McDougall (2003), recall that the latter is defined by \(\Theta = h_T/C_{p0}\), where \(h_T\) is potential enthalpy, and \(C_{p0}\) a constant reference value of specific heat pressure (McDougall, 2003). Potential enthalpy, on the other hand, is implicitly defined in terms of the classical enthalpy \(h = h(S, T, p)\) by:

\[
h_T(S, T, p, p_r) = h(S, \theta(S, T, p, p_r), p_r) = h_r(S, \theta),
\]  

(A.12)
e.g., Feistel (2005). To proceed, we need the differential of enthalpy, viz.,

\[
dh = \mu dS + T d\eta + \frac{dp}{\rho} = \left( \mu - T \frac{\partial \mu}{\partial T} \right) dS + C_p dT + \frac{dp}{\rho}
\]  

(A.13)
estimated at the reference pressure at which \(T = \theta\). As a result, we can write

\[
d\Theta = \frac{1}{C_{p0}} \left( \mu_r - \theta \frac{\partial \mu_r}{\partial \theta} \right) dS + \frac{C_{pr}}{C_{p0}} d\theta.
\]  

(A.14)

This expression can be inverted to provide \(d\theta\) as a function of \(d\Theta\) and \(dS\) as follows:

\[
d\theta = \frac{C_{p0}}{C_{pr}} d\Theta - \frac{1}{C_{pr}} \left( \mu_r - \theta \frac{\partial \mu_r}{\partial \theta} \right) dS,
\]  

(A.15)

which can be inserted into (A.9) to yield:

\[
d\rho = \left[ \rho_\beta + \rho_\alpha \frac{C_{p0}}{C_{pr}} \left( \mu_r - \theta \frac{\partial \mu_r}{\partial \theta} \right) \right] dS - \rho_\alpha \frac{C_{p0}}{C_{pr}} d\Theta + \frac{1}{c_s^2} dp.
\]  

(A.16)
It follows immediately that
\[ \alpha_\Theta = \left( \frac{TC_{p0}}{\partial C_p} \right) \alpha \] (A.17)
\[ \beta_\Theta = \beta_\theta + \frac{\alpha_T}{C_{pr}} \left( \mu_r - \theta \frac{\partial \mu_r}{\partial \theta} \right) \beta_\Theta = \beta \left[ 1 + \frac{\alpha T}{\beta C_p} \left( \frac{\mu_r}{\theta} - \frac{\partial \mu}{\partial T} \right) \right], \] (A.18)

which are the sought-for expressions for the haline contraction and thermal expansion coefficients in the \((S, \Theta, p)\) space. Note that the compressibility coefficient is the same regardless of whether \(\eta\), \(\theta\), or \(\Theta\) is used.

**APPENDIX B**

**Irreversible production of potential temperature and conservative temperature**

The purpose of this section is to provide explicit expressions for the irreversible production terms in the evolution equations for potential and conservative temperatures, i.e., to estimate the term \(\dot{\theta}_{irr}\) in the following equation:
\[ \frac{D \theta}{Dt} = \frac{1}{\rho} \nabla \cdot F_\theta + \dot{\theta}_{irr}, \] (B.1)

which is aimed at replacing Eq. (5) in the equations of motion, where \(\theta\) refers to either potential or conservative temperature (assuming then \(\rho\) to be a function of \((S, \theta, p)\) rather than \((S, \eta, p)\)). The sought-for results are obtained by invoking the energy conservation argument developed in section 2. To that end, we use Eqs. (A.8) and (A.15) to derive the following expressions for \(dS + T d\theta\), which are the first terms in the total differential for internal energy or enthalpy, in terms of \(d\theta\) and \(dS\) (and \(d\Theta\)) respectively:
\[ \mu dS + T d\eta = \left( \mu - T \frac{\partial \mu_r}{\partial \Theta} \right) dS + \frac{TC_{pr}}{\theta} d\theta, \] (B.2)
\[ \mu dS + T d\eta = \left( \mu - T \frac{\mu_r}{\theta} \right) dS + \frac{TC_{p0}}{\theta} d\Theta. \] (B.3)

Apart from differences in notations, these results are consistent with the terms within curly brackets in Eqs. (14) and (25) of McDougall (2003), which are associated with the expression of the first law in terms of \(\theta\) and \(\Theta\) respectively. By using these two expressions, and proceeding as in section 2, we can show that the conservation equation for the total energy can be written in either of the following form:
\[ \frac{\partial E}{\partial t} + \nabla \cdot F_E = \rho \left[ \frac{TC_{pr}}{\theta} \dot{\theta}_{irr} - F_\Theta \cdot \nabla \left( \mu - T \frac{\partial \mu_r}{\partial \Theta} \right) - F_\Theta \cdot \nabla \left( \frac{TC_{pr}}{\theta} \right) - E \right], \] (B.4)
\[ \frac{\partial E}{\partial t} + \nabla \cdot F_E = \rho \left[ \frac{TC_{p0}}{\theta} \dot{\Theta}_{irr} - F_S \cdot \nabla \left( \mu - T \frac{\mu_r}{\theta} \right) - F_\Theta \nabla \left( \frac{TC_{p0}}{\theta} \right) - E \right], \] (B.5)

where the expression for the energy flux \(F_E\) differs in each case, taking into account the fact that the flux of internal energy takes a different form depending on whether \(\theta\) or \(\Theta\) is used. As in section 2, we invoke the law of total energy conservation to require the right-hand sides of Eqs. (B.4) and (B.5) to vanish, yielding:
\[ \dot{\theta}_{irr} = \frac{\theta}{TC_{pr}} \left[ F_S \cdot \nabla \left( \mu - T \frac{\partial \mu_r}{\partial \Theta} \right) + F_\Theta \cdot \nabla \left( \frac{TC_{pr}}{\theta} \right) + E \right], \] (B.6)
\[ \dot{\Theta}_{irr} = \frac{\theta}{TCp0} \left[ F_S \cdot \Delta \left( \mu - \frac{T\mu_r}{\theta} \right) + F_\Theta \cdot \Delta \left( \frac{TCp0}{\theta} \right) + \varepsilon \right]. \]  

(B.7)

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