The capacitance of realistic ice particles

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The problem:

Growth/evaporation of ice particles by diffusion of vapour

\[ \frac{dm}{dt} = 4\pi C (\rho_s - \rho_\infty) \]

Houghton (1950)  
J. Met. 7 363

Mass transfer driven by difference in vapour density at ice surface compared to the background, controlled by effective radius C (capacitance)

- same equation for conduction of latent heat to/from crystal
- assumed constant surface vapour pressure (sat. value)
- constant surface T (thermal conductivity ice >> air)

Idea has been around for over 50 years  
Used universally in cloud schemes to convert between vapour & IWC  
but no-one knows what C is supposed to be!
Analytical results:

**Sphere**

\[ C = 0.5 \, D \]

**Cigar**

- complicated trig formulas

\[ C = D \times f(\text{aspect ratio}) \]

**Pancake**

- thin disc \( C = 0.32 \, D \)

**Circular cylinder** - analytic expansion, approx formula:

\[ C = 0.32(1 + 0.87A^{0.76})D \]

**But what about realistic ice particle shapes?**

*Table 1: Theoretical capacitances for simple shapes (McDonald 1963). Note that electrical capacitances in Farads are usually normalised as \( 4\pi \varepsilon C \), where \( \varepsilon \) is the permittivity of the surrounding medium in Fm\(^{-1}\).*

<table>
<thead>
<tr>
<th>Shape</th>
<th>Capacitance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere, radius ( r )</td>
<td>( C = r )</td>
</tr>
<tr>
<td>Thin disc, radius ( r )</td>
<td>( C = 2r/\pi )</td>
</tr>
<tr>
<td>Prolate spheroid: major,</td>
<td>( C = A/\ln[(a + A)/b] ), where ( A = \sqrt{a^2 - b^2} )</td>
</tr>
<tr>
<td>minor semi-axes ( a, b )</td>
<td></td>
</tr>
<tr>
<td>Oblate spheroid: major,</td>
<td>( C = ae/\sin^{-1}e ), where ( e = \sqrt{1 - c^2/a^2} )</td>
</tr>
<tr>
<td>minor semi-axes ( a, c )</td>
<td></td>
</tr>
</tbody>
</table>
1960s - metal models of ice shapes

- measure capacitance of model in a Faraday cage (a few pF)
- problem is short-out of field lines from connecting wire, systematic errors of as much as 40%
- best you can do is measure ratio cf. spheroid of same overall dimensions

\[
\begin{align*}
\text{C (model crystal, measured)} & \quad \text{assume} \quad \text{C (model crystal, actual)} \\
\text{C (model spheroid, measured)} & \quad = \quad \text{C (spheroid, theory)} \quad \text{know}
\end{align*}
\]

essentially, showed plate ≈ pancake, column ≈ cigar
2000s - numerical solution of Laplace equation


- simplified bullet-rosette shapes (smooth lobes)
- solve using finite-difference, calculate vapour flux $\rightarrow$ C
- expensive: computer time and lots of memory (have to discretise whole domain, and outer boundary condition is fixed far from particle)

← same procedure for columns

Chiruta & Wang (2005)

GRL 32 L05803
New Monte Carlo approach

Sample trajectories of individual water molecules

- sounds like a disaster, but find only need ~ $10^4$ trajectories to get a well sampled total flux (and \( \therefore \) C)

Water molecules represented by random walkers
- start by looking at the simplest possible method (slow)
- then use random walker statistics to make big improvements

Each trajectory is sampled independently, ie.
- almost no computer memory is needed,
- no need to discretise/approximate particle or surrounding space
- no problems with sharp corners & edges (singularities)
1. Start random walks on big sphere

2. Take tiny steps in random direction

3.a) escapes to infinity

3.b) hits the ice particle

Then $C = f \times R_\infty$ (f=fraction of walks that hit)

Problem: walkers spend most of time taking tiny steps far from the particle $\rightarrow$ very slow
Use random walk statistics to make big improvements

Trick #1: start random walks on a much closer sphere that *just* encloses ice particle

Flux of walkers coming from $R_\infty$ intersecting $R$ for the first time is uniformly distributed over surface.
Know this flux = $4\pi D\rho_\infty R$

But net flux = this - (walks which pass through $R$ but end up back at $R_\infty$) - so must track walkers back to $R_\infty$ if they don't hit the particle
Trick #2: random walks are isotropic!

Water molecule follows unobstructed random walk over linear distance $a$. What's the pdf of where it ends up? Uniform over surface of sphere, radius $a$

So outside $R$ no need to take tiny steps - make as big as distance back to $R$!

Inside $R$ need to be slightly more careful - take steps as big as distance to closest point on particle surface.

Count a hit if comes within $\delta$ of ice surface ($\delta < 0.1\%$ of smallest side of particle)

Then $C = f \times R$
MUCH BETTER!

Simplest example: **hexagonal column**
- takes ~1 minute on desktop PC to accuracy of 1%

Metal model experiments - columns and plates should be similar to spheroids of same overall dimensions

Numerical work (Chiruta & Wang) suggested columns ~20-30% higher than spheroids.

Intuitively expect results similar to circular cylinder
Confirms metal-model experiments. Chiruta & Wang are overestimates (can construct analytical bounds to confirm this.)
Bullet rosettes

Single bullets look ~ 10% less than column of same overall dimensions
2 arm rosette ~ 15% less than column

4 & 6 arm rosettes:
Stellars & dendrites

- Stars lower $C$ than plate
- Extra branches pushes $C$ up cf. star
- Both approach solid plate for fat arms
Aggregates

Simplified simulation of aggregation process

Simulations produce large sample of aggregate shapes

Geometry is consistent with in-situ observations, density $\sim D_{\text{max}}^{-1}$
Aggregate results

Results for different monomer crystals all approach value of \( C=0.25D_{\text{max}} \)

(both columns and bullet-rosettes for range of aspect ratios)

Good agreement with in-situ observations of sublimating aggregates in anvil clouds \( (C=0.23D_{\text{max}} \text{ Field et al 2007 JAS in press}) \)

Seems surprising \( C/D_{\text{max}} \rightarrow \text{constant} \) because of fall off in density with increasing size, but essentially because water molecules very efficient at exploring the space \((fractal \text{ path } d_f=2)\)
Screening in cloud chamber expts

Crystals grown on glass fibre

Hard to control nucleation - crystals grow near each other.

Do these crystals compete for vapour?

Bailey & Hallet 2004 - measured growth rates and concluded capacitance much too high

- should we worry about comparing measured growth rates with theory?

**YES** - factor of 3 difference!
Summary by habit
realistic aspect ratios

all values in the range $C/D_{max}=0.25 \pm 30\%$
Implications for cloud parameterisation

- Most cloud parameterisations come from aircraft data (measure Dmax)
- Detailed results on C could be used in models which explicitly predict habit (eg. Woods et al JAS 2007)
- More general rule of thumb $C = 0.25D_{max}$ useful for cruder schemes where don’t want to worry about the exact crystal shape
- Pins down C quite well so can ascribe model/obs discrepancies to other things (inadequate resolution of RH gradients, mass-size relationship, fall speed etc)

- For low supersat/small xtals worry about accommodation co-efficient (reduces growth/evap rate)
- For bigger crystals ventilation effect important
  - very little relevant data on this!
- Possible that can use C to parameterise ventilation effect too…
$X_c = 0.86 \, \text{Re}^{1/2}$

Use C as length scale in Re - correlates data to within $\sim 20\%$

cf. Hall & Pruppacher use $L^* = \text{surface area/perimeter of projection}$ ... hard to estimate from aircraft data and dubious for complex/rough shapes
Summary

- Simple & fast method to calculate ice particle capacitance
- Arbitrary shape, sharp edges & corners OK
- Could help experimentalists to pin down accommodation co-eff
- But need to worry about screening in cloud chamber experiments!
- Rule of thumb C~0.25 $D_{max}$ for most habits including aggregates
- C good length scale to correlate ventilation data

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Westbrook, Hogan & Illingworth JAS in press
‘The capacitance of pristine ice crystals and aggregate snowflakes’

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