Time-stepping, numerical stability, dispersion and conservation

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Sources

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Contents

Chapter 1: Time-Stepping

Any system of evolution equations can be written as

$$\frac{dy}{dt} = F(y)$$

where y is a list of all dependent variables at all points in space and F describes how they all evolve. There are thousands of time-stepping schemes. Eg:

		Explicit/ Implicit	Order of accuracy	Multi stage/ step/neither
Forward Euler	$y^{(n+1)} = y^{(n)} + \Delta t F(y^{(n)})$	E	1	neither
Backward Euler	$y^{(n+1)} = y^{(n)} + \Delta t F(y^{(n+1)})$	I	1	neither
Trapezoidal (Crank-Nicholson)	$y^{(n+1)} = y^{(n)} + \frac{\Delta t}{2}(F(y^{(n)}) + F(y^{(n+1)}))$	I	2	neither
Forward-backward	$y' = y^{(n)} + \Delta t F(y^{(n)})$ $y^{(n+1)} = y^{(n)} + \Delta t F(y')$	E	1/2	stage
Leapfrog (centred in time)	$y^{(n+1)} = y^{(n-1)} + 2\Delta t F(y^{(n)})$	E	2	step

Explicit – uses values from previous time-steps to define the values at the new time-level.

Implicit – uses values at time level $n+1$ and possibly other time levels to define the values at time-level $n+1$.

Multi-step – uses values from more that 2 time-levels.

Multi-stage (Runge-Kutta) – Calculates intermediate values in between levels n and $n+1$.

		Explicit/ Implicit	Order of accuracy	Multi stage/ step
RK4	$k_1 = \Delta t F(y^{(n)}),\ k_2 = \Delta t F(y^{(n)} + \frac{1}{2}k_1)$ $k_3 = \Delta t F(y^{(n)} + \frac{1}{2}k_2),\ k_4 = \Delta t F(y^{(n)} + k_3)$ $y^{(n+1)} = y^{(n)} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	E	4	stage
Adams-Bashforth	$y^{(n+1)} = y^{(n)} + \frac{\Delta t}{2}\left(3F(y^{(n)}) - F(y^{(n-1)})\right)$	E	2	step
BDF2	$y^{(n+1)} = \frac{4}{3}y^{(n)} - \frac{1}{3}y^{(n-1)} + \Delta t \frac{2}{3}F(y^{(n+1)})$	I	2	step

Advantage of Multi-step over Multi-stage

Fewer function evaluations needed to achieve the same order accuracy

How can an Implicit scheme be used?

Re-arrange backward Euler: $y^{(n+1)} - \Delta t F(y^{(n+1)}) = y^{(n)}$

Define a new function: $(I - \Delta t F)$ where I is the identity

$y^{(n+1)} = (I - \Delta t F)^{-1} y^{(n)}$

F needs to be a linear function so that $(I - \Delta t F)$ is an invertible matrix

Advantage of Explicit over Implicit

No need to invert a big matrix $\therefore$ cheaper per time-step

Advantage of Implicit? ... Advantage of Multi-stage? ...

1.1 Stability Analysis of Leapfrog

To analyse the stability of a time-stepping scheme for solving a wave or advection equation, we analyse how the scheme behaves for the 1D oscillation equation:

$$\frac{dy}{dt} = i\kappa y \tag{1.1}$$

where $i = \sqrt{-1}$ so that the leapfrog scheme becomes

$$y^{(n+1)} = y^{(n-1)} + i2\Delta t\,\kappa y^{(n)}. \tag{1.2}$$

We define an amplification factor, A, such that:

$$y^{(n+1)} = Ay^{(n)},\ y^{(n)} = Ay^{(n-1)},\ y^{(n+1)} = A^2y^{(n-1)}. \tag{1.3}$$

The scheme will be stable if $|A| \le 1$

Substitute the amplification factors in eqn (1.3) into eqn(1.2) and rearrange to find A and $|A|$:

$$\begin{aligned}
A^2y^{(n-1)} &= y^{(n-1)} + i2\Delta t\,\kappa A y^{(n-1)} \\
\implies A^2 - i2\Delta t\,\kappa A - 1 &= 0 \\
\implies A &= i\kappa\Delta t \pm \sqrt{1-\kappa^2\Delta t^2} \\
\implies |A| &\begin{cases} = 1 & \text{if } \kappa\Delta t \le 1 \\ > 1 & \text{if } \kappa\Delta t > 1 \end{cases}
\end{aligned}$$

So leapfrog is stable for $\Delta t \le 1/\kappa$. But the existence of two possible solutions for A means that spurious solutions - computational modes - exist which can contaminate the solution.

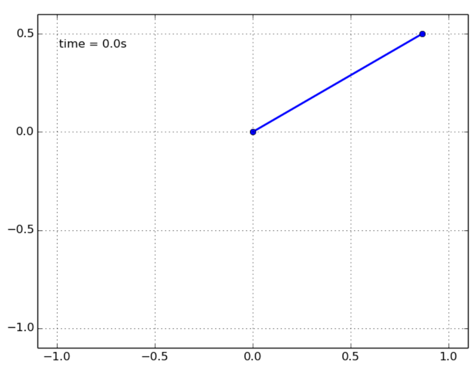
1.1.1 Simulation of a Damped Pendulum

Angle of pendulum, θ, satisfies

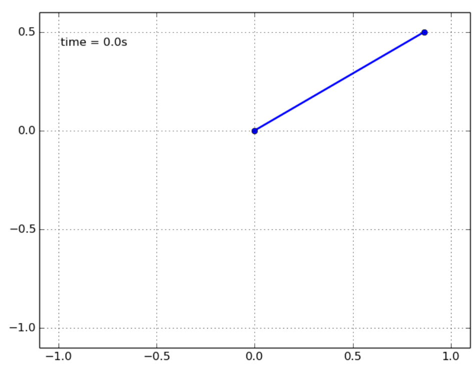
$$\frac{d^2\theta}{dt^2} + \frac{\alpha}{L}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0$$

where $L = 1$, $g = 9.81$, $\alpha = 0.03$

Leapfrog



RK4



- The leapfrog simulation jumps between the physical mode and the computational mode
- Can be controlled with a filter [see eg Williams, 2009]

1.1.2 Stability Analysis of Euler Forward and Backward

Exercise We will analyse the stability of Euler forward and Euler backward for both the oscillation equation (to mimic wave equations and advection) and for the equation that simulations exponential decay (to mimic diffusion):

$$\frac{dy}{dt} = -\kappa y$$

Choose one of the schemes and one of the differential equations. So that we get answers for all four possibilities, make sure that your choices are different from 3 of your neighbours.

Stability Constraints for Euler forward and backward

		Euler forward	Euler backward
		$y^{(n+1)} = y^{(n)} + \Delta t F(y^{(n)})$	$y^{(n+1)} = y^{(n)} + \Delta t F(y^{(n+1)})$
Oscillation equation	$F(y) = i\kappa y$	unstable $\forall\ \Delta t$	stable $\forall\ \Delta t$
Exponential decay	$F(y) = -\kappa y$	stable for $\Delta t \le 2/\kappa$	stable $\forall\ \Delta t$

1.1.2.1 Solutions

1. Euler forward for the oscillation equation.
   $A = 1 + i\Delta t\kappa \implies |A|^2 = 1 + \kappa^2\Delta t^2 > 1\ \forall \Delta t > 0,\ \kappa > 0$
2. Euler backward for the oscillation equation.
   $A = 1 + i\Delta t\kappa A \implies A = 1/(1 - i\Delta t\kappa) \implies |A|^2 = 1/(1 + \kappa^2\Delta t^2) < 1\ \forall \Delta t > 0,\ \kappa > 0$
3. Euler forward for the exponential decay.
   $A = 1 - \Delta t\kappa \implies |A| \le 1 \iff 1 - \Delta t\kappa \ge -1 \iff \Delta t\kappa \le 2$
4. Euler backward for the exponential decay.
   $A = 1 - \Delta t\kappa A \implies A = 1/(1 + \Delta t\kappa) \implies |A| < 1\ \forall \Delta t > 0,\ \kappa > 0$

1.2 Summary of Advantages of Different Types of Time-Stepping Schemes

Explicit

Cheap to compute

Implicit

Stable with large time-steps

Multi-step

Cheap since fewer function evalulations

Multi-stage

No computational modes

Chapter 2: Modelling Wave Equations

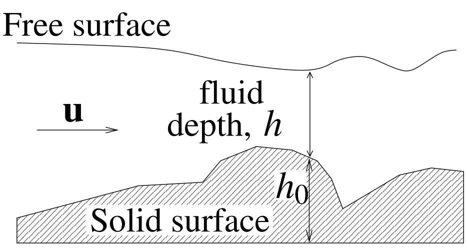
Many of the processes in the atmosphere are represented by the shallow water equations (SWE). The assumptions needed to derive the SWE are:

- Horizontal length scale >> vertical length scale
- Very small vertical velocities

Depth integrate the Navier-Stokes equations over orography to give the SWE:

$$\frac{D\mathbf{u}}{Dt} = -2\Omega \times \mathbf{u} - g\nabla(h + h_0) + \mu_u \nabla^2 \mathbf{u} \tag{2.1}$$

$$\frac{Dh}{Dt} + h\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$



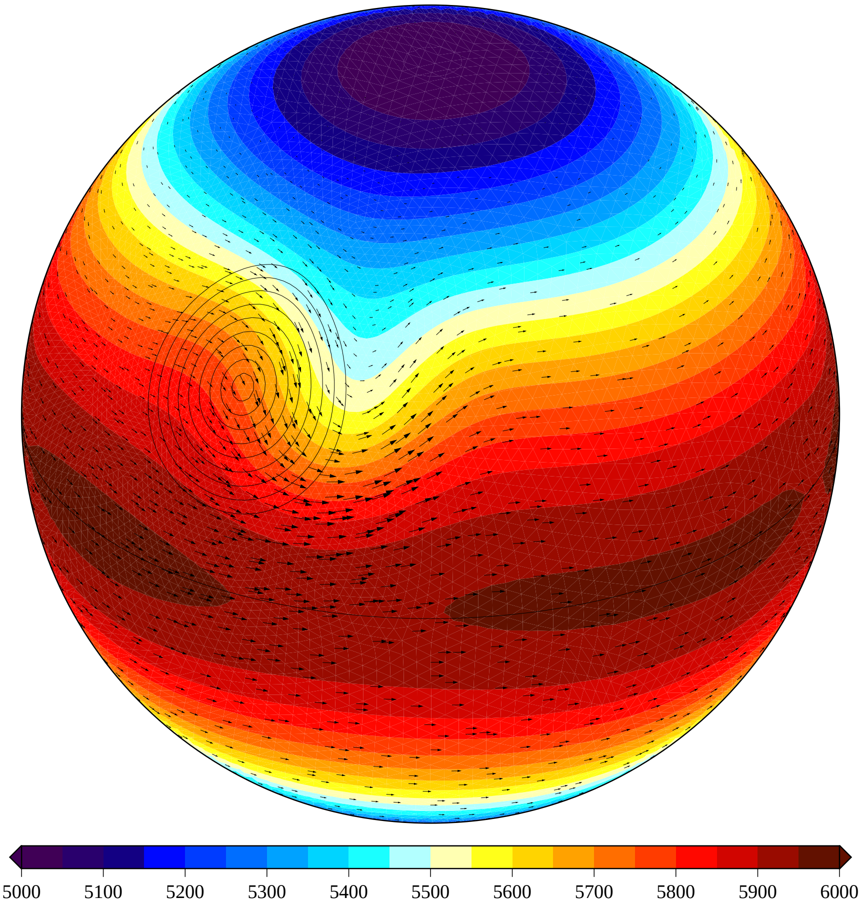
where

$\mathbf{u}$	Depth intetraged wind vector	g	Acceleration due to gravity
t	Time	∇	Gradients in the horizontal
Ω	Rotation rate of planet	h_0	Height of the bottom topography
h	Fluid depth	μ_u	Diffusion of momentum

Exercise: What are the meaning of the terms of the momentum equation of the SWE?

2.1 Simulations of the SWE on the surface of a sphere

The shallow-water equations can be solved on the surface of a sphere with **u** being the horizontal wind (ignoring updrafts and downdrafts) and h being the depth of a layer of atmosphere. The results look similar to large-scale atmospheric circulation. The vectors show **u**, the black contours show a mountain (h_0) and colours show $h+h_0$.



2.2 Processes Represented by the SWE

Which of these processes are represented by the SWE and which are only represented by the full NS equations?

Horizontal advection	SWE	Acoustic waves	NS
Vertical advection	NS	Coriolis	SWE
Gravity waves	SWE	Diffusion	SWE
Rossby waves	SWE	Heat transport	NS
Adiabatic expansion	NS	Atmospheric convection	NS
Geostrophic balance	SWE	Geostrophic turbulence	SWE

Component Form of the SWE

Assuming $\mathbf{u} = (u, v, 0)^T$ and $2\Omega = (0, 0, f)^T$, equations (2.1) and (2.2) written in component form are:

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= fv-g\frac{\partial(h+h_0)}{\partial x}+\mu_u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -fu-g\frac{\partial(h+h_0)}{\partial y}+\mu_u\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\end{aligned}$$

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}+h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \quad \text{or} \quad \frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}+\frac{\partial hv}{\partial y}=0$$

2.2.1 Linearised SWE

In order to find analytic solutions and to analyse numerical methods, we linearise the SWE. Assume:

- $\mathbf{u} = (u, v, 0)^T$ is small
- $2\Omega = (0, 0, f)^T$
- $h = H + h'$ where H is uniform in space and time and h' is small
- the product of two small variables is ignored (even if one or both are inside a differential)
- h_0 and μ_u are ignored

This gives the following equations for u,v and h' expressed in terms of f (rather than Ω):

$$\frac{\partial u}{\partial t} = fv - g\frac{\partial h'}{\partial x} \tag{2.3}$$

$$\frac{\partial v}{\partial t} = -fu - g\frac{\partial h'}{\partial y} \tag{2.4}$$

$$\frac{\partial h'}{\partial t} = -H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{2.5}$$

2.3 Analytic Solultion

Ignoring Coriolis, the linearised SWE have wave-like solutions – ***gravity waves***. In 1d these are:

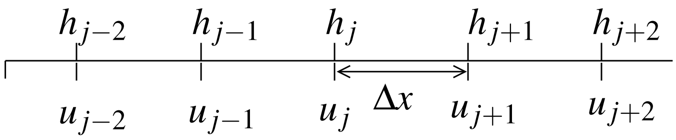
$$h' = \mathbb{H}\, e^{ikx}\, e^{\pm ikt\sqrt{gH}} \tag{2.6}$$

$$u = \pm\sqrt{g/H}\,\mathbb{H}\, e^{ikx}\, e^{\pm ikt\sqrt{gH}} \tag{2.7}$$

for any constant $\mathbb{H}$. So waves with wavenumber k in space oscillate with frequency $k\sqrt{gH}$ and the wave speed is ... $\sqrt{gH}$ (so gravity waves are non-dispersive).

2.4 Unstaggered Forward-Backward (1d A-grid FB)

As there are two equations that depend on each other, it is quite natural to solve them using forward-backward time-stepping – forward for u and backward for h. We will also start by assuming that h and u are defined at the same spatial positions (this is called co-located, unstaggered or A-grid) and we will use centred spatial discretisation:



$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} \rightarrow \qquad \frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t} = -g\frac{h_{j+1}^{(n)} - h_{j-1}^{(n)}}{2\Delta x} \tag{2.8}$$

$$\frac{\partial h}{\partial t} = -H\frac{\partial u}{\partial x} \rightarrow \qquad \frac{h_j^{(n+1)} - h_j^{(n)}}{\Delta t} = -H\frac{u_{j+1}^{(n+1)} - u_{j-1}^{(n+1)}}{2\Delta x} \tag{2.9}$$

where $x_j = j\Delta x$, $t^{(n)} = n\Delta t$, $h_j^{(n)} = h(x_j, t^{(n)})$ and $u_j^{(n)} = u(x_j, t^{(n)})$.

2.4.1 Von-Neumann Stability Analysis

We can find the stability limits and dispersion relation for the numerical scheme given in section 2.4 (1d A-grid FB) using von-Neumann stability analysis.

To calculate an amplification factor, A, for each wavenumber, k, we assume wave-like solutions for h and u:

$$h_j^{(n)} = \mathbb{H}\, A^n\, e^{ikj\Delta x} \tag{2.10}$$

$$u_j^{(n)} = \mathbb{U}\, A^n\, e^{ikj\Delta x} \tag{2.11}$$

for some constants $\mathbb{H}$ and $\mathbb{U}$. Substituting these into (2.8) and (2.9) and defining the Courant number $c = \dfrac{\sqrt{gH}\Delta t}{\Delta x}$ leads to: (workings on lecturer notes)

$$A = 1 - \frac{c^2}{2}\sin^2 k\Delta x \pm \frac{ic}{2}\sin k\Delta x\sqrt{4 - c^2\sin^2 k\Delta x} \tag{2.12}$$

There are two solutions for A but this is correct because there are also two analytic solutions to the equations (because of the $\pm$ in the analytic solution). For $|c| \leq 2$ this gives $|A|^2 = 1$ so the scheme is stable and undamping for sufficiently small time steps. However for $|c| > 2$ we have:

$$|A|^2 = \left(1 - \frac{c^2}{2}\sin^2 k\Delta x \pm \frac{c}{2}\sin k\Delta x\sqrt{c^2\sin^2 k\Delta x - 4}\right)^2$$

which can be greater than 1 and so the scheme is unstable for $|c| > 2$ where $c = \sqrt{gH}\Delta t/\Delta x$. So this scheme is conditionally stable. Stable for $c \leq 2$.

Von-Neumann Stability Analysis of Unstaggered Forward-Backward

Substitute eqns (2.10) and (2.11) into eqns (2.8) and (2.9):

$$\frac{\mathbb{U}\,A^{n+1}\ e^{ikj\Delta x} - \mathbb{U}\,A^{n}\ e^{ikj\Delta x}}{\Delta t} = -g\frac{\mathbb{H}\,A^{n}\ e^{ik(j+1)\Delta x} - \mathbb{H}\,A^{n}\ e^{ik(j-1)\Delta x}}{2\Delta x}$$
$$\frac{\mathbb{H}\,A^{n+1}\ e^{ikj\Delta x} - \mathbb{H}\,A^{n}\ e^{ikj\Delta x}}{\Delta t} = -H\frac{\mathbb{U}\,A^{n+1}\ e^{ik(j+1)\Delta x} - \mathbb{U}\,A^{n+1}\ e^{ik(j-1)\Delta x}}{2\Delta x}.$$

Simplify by substituting in the Courant number, $c = \sqrt{gH}\Delta t/\Delta x$, cancelling $A^n e^{ikj\Delta x}$ and combining $\mathbb{H}$ and $\mathbb{U}$:

$$A - 1 = -\frac{c}{2}\sqrt{\frac{g}{H}} \sim \frac{\mathbb{H}}{\mathbb{U}}\,A\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$
$$\frac{\mathbb{H}}{\mathbb{U}}(A-1) = -\frac{c}{2}\sqrt{\frac{H}{g}}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right).$$

$\mathbb{H}/\mathbb{U}$ can be eliminated and we can substitute in $e^{\pm ik\Delta x} = \cos k\Delta x \pm i\sin k\Delta x$ to get an expression for the amplification factor, A, in real and imaginary parts:

$$\frac{\mathbb{H}}{\mathbb{U}} = -ic\sqrt{\frac{H}{g}}\,\frac{1}{A-1}\sin k\Delta x \qquad (2.13)$$
$$\Longrightarrow A - 1 = -c^2\frac{A}{A-1}\sin^2 k\Delta x \qquad (2.14)$$
$$\Longrightarrow A^2 - A(2 - c^2\sin^2 k\Delta x) + 1 = 0 \qquad (2.15)$$
$$\Longrightarrow A = \frac{2 - c^2\sin^2 k\Delta x \pm \sqrt{(2 - c^2\sin^2 k\Delta x)^2 - 4}}{2} \qquad (2.16)$$
$$\Longrightarrow A = 1 - \frac{c^2}{2}\sin^2 k\Delta x \pm \frac{ic\sin k\Delta x}{2}\sqrt{4 - c^2\sin^2 k\Delta x^2}. \qquad (2.17)$$

In order to find the stability of this scheme we need to find the magnitude of the amplification factor, $||A|| = \sqrt{AA^*}$ and compare this to one. For the above amplification factor, if $c^2\sin^2 k\Delta x < 4\ \forall\ k\Delta x$ then the value inside the square root is positive and we can calculate:

$$\begin{aligned}||A||^2 &= \left(1 - \frac{c^2}{2}\sin^2 k\Delta x\right)^2 + \frac{c^2\sin^2 k\Delta x}{4}\left(4 - c^2\sin^2 k\Delta x^2\right)\\ &= 1 - c^2\sin^2 k\Delta x + \frac{c^4}{4}\sin^4 k\Delta x + c^2\sin^2 k\Delta x - \frac{c^4}{4}\sin^4 k\Delta x\\ &= 1.\end{aligned}$$

So for $c < 2$ this scheme is stable and does not damp waves of any frequency. However for $c > 2$, $||A||$ can be greater that 1 for some values of $k\Delta x$ so the scheme is unstable.

2.4.2 Dispersion of Unstaggered Forward-Backward (1d A-grid FB)

A reminder of the amplification factor for this method:

$$A = 1 - \frac{c^2}{2}\sin^2 k\Delta x \pm \frac{ic}{2}\sin k\Delta x\sqrt{4 - c^2\sin^2 k\Delta x}.$$

The argument of A gives us the wave frequency as a function of wavenumber:

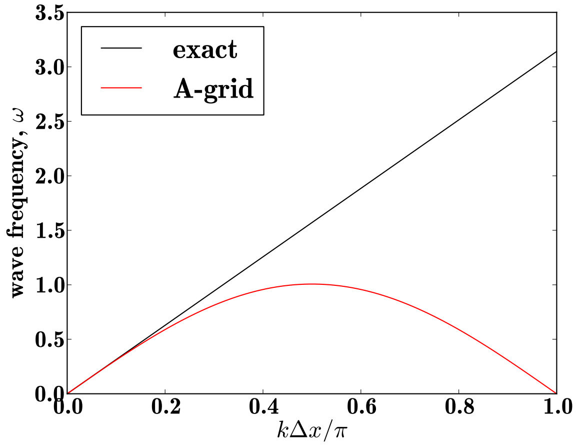
$$\omega = \tan^{-1}\frac{\frac{c}{2}\sin k\Delta x\sqrt{4 - c^2\sin^2 k\Delta x}}{1 - \frac{c^2}{2}\sin^2 k\Delta x} \tag{2.18}$$

This can be simplified by assuming that $\frac{c}{2}\sin k\Delta x = \sin\alpha$ to give:

$$\omega = \pm 2\alpha = \pm 2\sin^{-1}\left(\frac{c}{2}\sin k\Delta x\right) \tag{2.19}$$

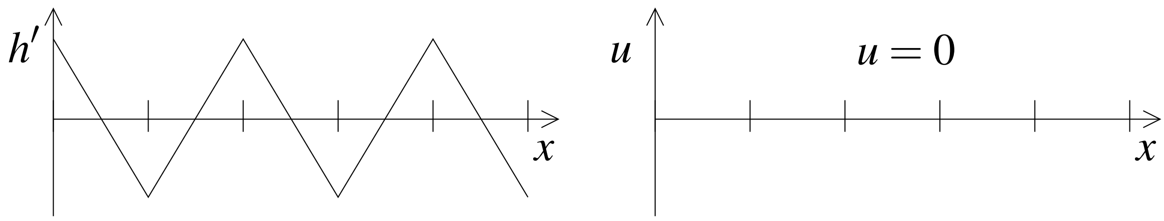
This is the A-grid dispersion relation:

Grid-scale gravity waves ($k\Delta x/\pi = 1$) have zero frequency! This is highly unrealistic.



2.5 Problems with Co-location of h and u

Consider the following initial conditions of the linearised non-rotating SWE:



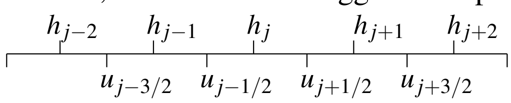
Questions:

1. How do you expect the real solution of the linearised SWE to evolve?
   High-frequency waves will be generated that propagate in both directions. The solution will oscillate between having non-zero h' and non-zero u.

2. How will the solution of the 1d A-grid FB scheme evolve?
   The solution will not change after initialisation. The grid-scale wave in h' will remain. No non-zero u will be generated.

2.6 Staggered Forward-Backward (1d C-grid FB)

So that gradients of h can be calculated where u is located and gradients of u can be calculated where h is located, h and u can be staggered in space:

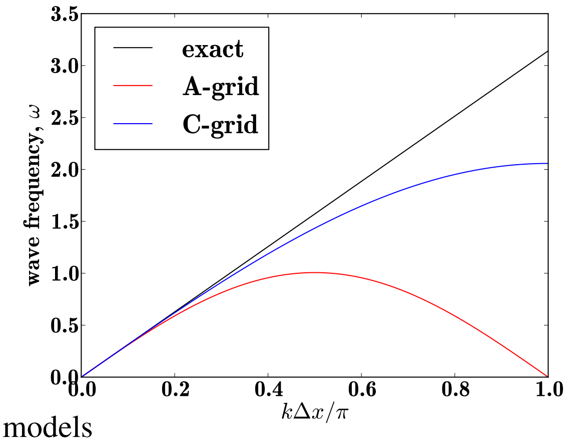


Using centered, 2-point spatial differences and forward-backward in time gives:

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} \quad \rightarrow \quad \frac{u^{(n+1)}_{j+1/2} - u^{(n)}_{j+1/2}}{\Delta t} = -g\frac{h^{(n)}_{j+1} - h^{(n)}_{j}}{\Delta x} \tag{2.20}$$

$$\frac{\partial h}{\partial t} = -H\frac{\partial u}{\partial x} \quad \rightarrow \quad \frac{h^{(n+1)}_{j} - h^{(n)}_{j}}{\Delta t} = -H\frac{u^{(n+1)}_{j+1/2} - u^{(n+1)}_{j-1/2}}{\Delta x} \tag{2.21}$$

Von-Neumann stability analysis gives:



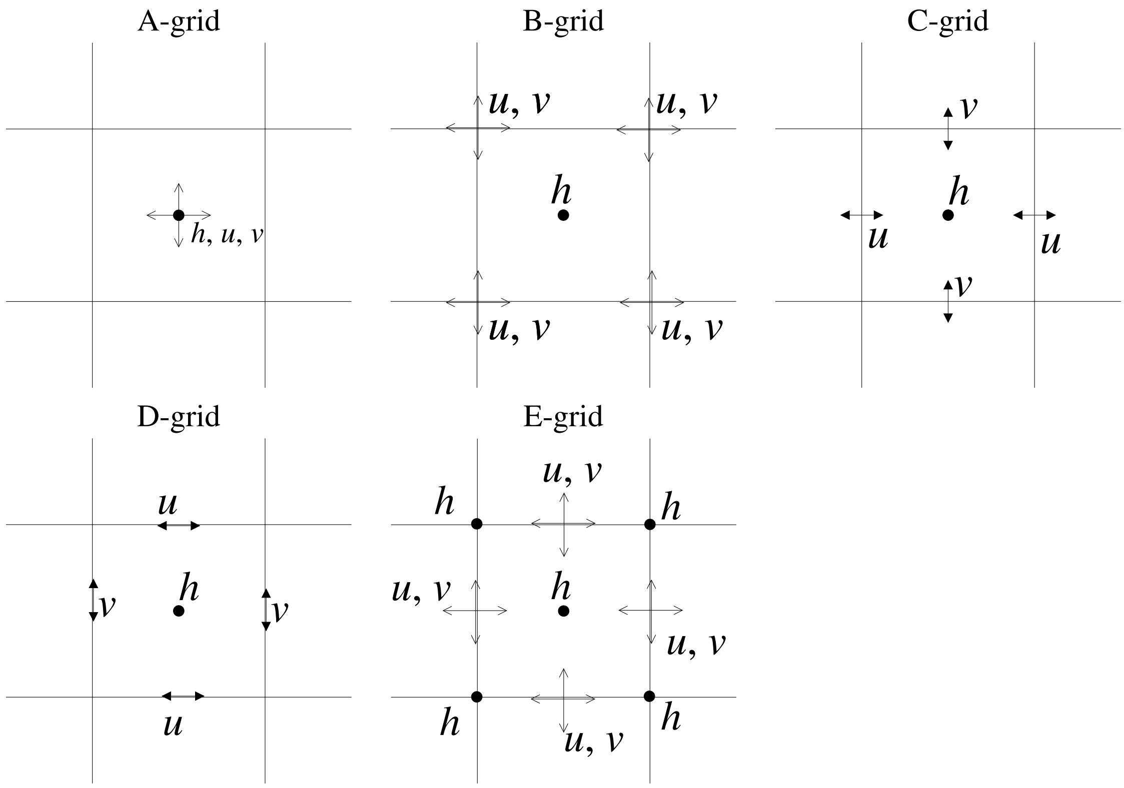
- $|A| = \begin{cases} 1 & \text{for } |c| \le 1 \\ > 1 & \text{for } |c| > 1 \text{ for some } k\Delta x \end{cases}$

  $\therefore$ neutrally stable for $|c| \le 1$
- Dispersion relation:

  $\omega = \pm 2\alpha = \pm 2\sin^{-1}\left(c \sin\frac{k\Delta x}{2}\right)$
- $\therefore$ the C-grid is dispersive
- grid-scale waves propagate too slowly
- C-grid widely used in atmosphere and ocean models
- What about in 2d?

2.7 Arakawa Grids

In two dimensions, there are more possibilities for where the prognostic variables are located:

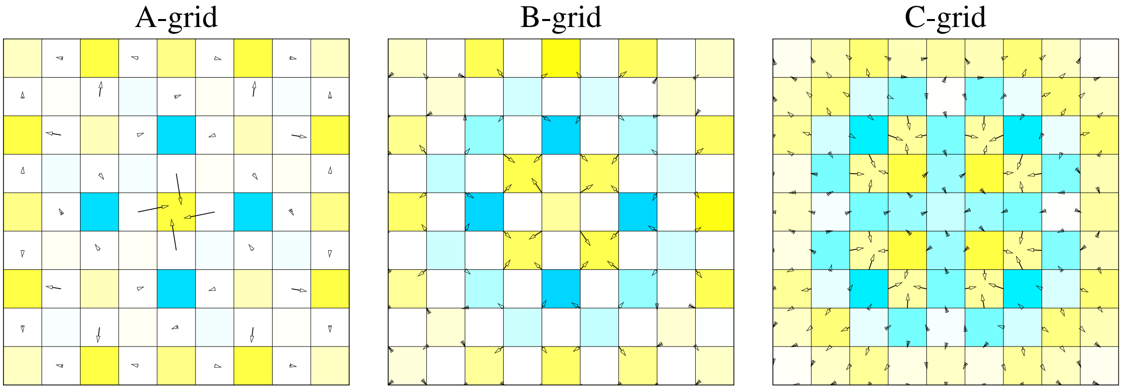


2.8 Linearised Shallow-Water Equations on Arakawa Grids

The linearised SWE with rotation are:

$$\partial \mathbf{u}/\partial t = -2\Omega \times \mathbf{u} - g\nabla h$$
$$\partial h'/\partial t + H\nabla \cdot \mathbf{u} = 0$$

- The linearised SWE are solved numerically on Arakawa A, B and C grids, starting from initial conditions consisting of zero velocity and zero h' everywhere except a positive h' in one central grid-box
- The colours show h' in the grid boxes. Red/yellow positive, blue negative, white zero



2.9 Discussion Question

For solving the 2d, linearised rotating SWE (eqns 2.3-2.5) what are the advantages and disadvantages of the different grids? Which terms or which balances between terms will be represented accurately by different grids?

2.10 Exercise

We have found that the numerical methods for solving the shallow water equations using forward-backward time-stepping have time-step restrictions based on the Courant number (defined with respect to the gravity wave speed). In the atmosphere, gravity waves can travel very quickly, up to about 300m/s - nearly as fast as acoustic waves. Complete models of the compressible atmosphere also support acoustic waves. We do not want the time-step of our models to be constrained by these very fast waves. Therefore, often, models are semi-implicit, which means that fast waves (such as acoustic and gravity waves) are treated implicitly whereas slow processes (such as advection and Coriolis) are treated explicitly. An implicit, co-located finite difference method for the one-dimensional, linearised, non-rotating shallow water equations is:

$$\frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t} = -g\frac{h_{j+1}^{(n+1)} - h_{j-1}^{(n+1)}}{2\Delta x} \tag{2.22}$$

$$\frac{h_j^{(n+1)} - h_j^{(n)}}{\Delta t} = -H\frac{u_{j+1}^{(n+1)} - u_{j-1}^{(n+1)}}{2\Delta x} \tag{2.23}$$

with the usual notation.

1. Use von-Neumann stability analysis to find the time-step restrictions of this scheme.
2. What problems does this scheme have in comparison to a staggered (or C-grid) scheme?

Answers to Exercise 2.10

1. Substituting $h_j^{(n)} = \mathbb{H}\, A^n\, e^{ikj\Delta x}$ and $u_j^{(n)} = \mathbb{U}\, A^n\, e^{ikj\Delta x}$ into the equations for the scheme gives:

$$\begin{aligned}
\frac{\mathbb{U}\, A^{n+1}\, e^{ikj\Delta x} - \mathbb{U}\, A^n\, e^{ikj\Delta x}}{\Delta t} &= -g\frac{\mathbb{H}\, A^{n+1}\, e^{ik(j+1)\Delta x} - \mathbb{H}\, A^{n+1}\, e^{ik(j-1)\Delta x}}{2\Delta x} \\
\frac{\mathbb{H}\, A^{n+1}\, e^{ikj\Delta x} - \mathbb{H}\, A^n\, e^{ikj\Delta x}}{\Delta t} &= -H\frac{\mathbb{U}\, A^{n+1}\, e^{ik(j+1)\Delta x} - \mathbb{U}\, A^{n+1}\, e^{ik(j-1)\Delta x}}{2\Delta x}.
\end{aligned}$$

We can simplify by substituting in the Courant number, $c = \sqrt{gH}\Delta t/\Delta x$, cancelling $A^n e^{ikj\Delta x}$ and combining $\mathbb{H}$ and $\mathbb{U}$:

$$\begin{aligned}
A - 1 &= -\frac{c}{2}\sqrt{\frac{g}{H}} \sim \frac{\mathbb{H}}{\mathbb{U}}\, A\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) \\
\frac{\mathbb{H}}{\mathbb{U}}(A-1) &= -\frac{c}{2}\sqrt{\frac{H}{g}}\, A\left(e^{ik\Delta x} - e^{-ik\Delta x}\right).
\end{aligned}$$

$\mathbb{H}/\mathbb{U}$ can be eliminated and we can substitute in $e^{\pm ik\Delta x} = \cos k\Delta x \pm i\sin k\Delta x$ to get an

expression for the amplification factor, A, in real and imaginary parts:

$$\frac{\mathbb{H}}{\mathbb{U}} = -ic\sqrt{\frac{H}{g}}\,\frac{A}{A-1}\sin k\Delta x \tag{2.24}$$

$$\Longrightarrow A - 1 = -c^2\frac{A^2}{A-1}\sin^2 k\Delta x \tag{2.25}$$

$$\Longrightarrow A^2\left(1 + c^2\sin^2 k\Delta x\right) - 2A + 1 = 0 \tag{2.26}$$

$$\Longrightarrow A = \frac{1 \pm \sqrt{1 - \left(1 + c^2\sin^2 k\Delta x\right)}}{1 + c^2\sin^2 k\Delta x} \tag{2.27}$$

$$\Longrightarrow A = \frac{1 \pm ic\sin k\Delta x}{1 + c^2\sin^2 k\Delta x}. \tag{2.28}$$

In order to find the stability of this scheme we need to find the amplitude of the amplification factor:

$$\begin{aligned}
||A||^2 &= AA^* \\
&= \frac{1 + c^2\sin^2 k\Delta x}{\left(1 + c^2\sin^2 k\Delta x\right)^2} \\
&= \frac{1}{1 + c^2\sin^2 k\Delta x}.
\end{aligned}$$

Thus $||A||^2 \le 1\ \forall\ c$ and $\forall\ k\Delta x$. Thus this implicit scheme is unconditionally stable (stable for all time-steps).

2. This co-located scheme has u and h stored at the same locations and spatial gradients are always calculated centred on a variable, missing out the variable in the middle.

Consequently, grid-scale oscillations do not influence the gradients and hence grid-scale oscillations do not lead to changes in u or h. In the linear shallow-water equations, all waves should propagate with speed $\sqrt{gH}$ but with this scheme, grid-scale waves (of wave-length $2\Delta x$) do not propagate all. Thus the scheme suffers from a spurious, stationary computational mode. This can also be seen by calculating the dispersion relation for the scheme, which shows that waves of length $2\Delta x$ are stationary. This problem is solved by using a staggered grid, with u and h stored at locations off-set from each other by $\Delta x/2$. For the shallow-water equations, it is necessary to calculate $\partial h/\partial x$ where u is stored and to calculate $\partial u/\partial x$ where h is stored. Consequently we will calculate $\partial h/\partial x$ at the mid-points between storage locations for h and the same for u. Thus the calculations of $\partial h/\partial x$ and $\partial u/\partial x$ will not miss out the central points; they will be compact. Any oscillations in u or h will lead to non-zero spacial gradients and will hence lead to changes in u and h. Therefore there is no stationary computational mode.

Chapter 3: Semi-Implicit Time-Stepping

Recap Some Advantages (and implied disadvantages) of aspects of time-stepping

Implicit	**Explicit**
Stable for Courant number $>> 1$	Cheap

Short time-steps	**Long time-steps**
Accurate	Cheap

Modelling Considerations

We can afford this setup for our model:

Δx, Δy	10km
Δz	200m
Δt	1 minute

The atmosphere has the following speeds:

Acoustic wave speed	340m/s
Gravity wave speed	$\leq$300m/s
Horizontal wind	$\leq$80m/s
Vertical wind	$\leq$1m/s

What Courant numbers will these speeds and resolutions lead to? What are the implications for choices of explicit and implicit time-stepping (assuming that explicit schemes are typically stable for Courant numbers less than one)?

	Maximum Courant number	**Implication**
Horizontal acoustic	$340 \times 60/10^3 \approx 2$	Treat horizontal acoustic waves implicitly, use a shorter time-step or use sub-stepping (split-explicit)
Vertical acoustic	$340 \times 60/200 \approx 100$	Vertically propaganing acoustic waves must be treated implicitly
Gravity wave	$300 \times 60/10^3 \approx 2$	Treat gravity waves implicitly, use a shorter time-step or use sub-stepping (split-explicit)
Horizontal wind	$80 \times 60/10^3 \approx 0.5$	Can be treated explicitly (unless you are on a lat-lon grid)
Vertical Wind	$1 \times 60/200 \approx 0.3$	Can be treated explicitly as long as you do not try to resolve convection or other processes which will increase the maximum vertical wind.

Treating gravity and acoustic wave implicitly and advection (and possibly Coriolis) explicitly is called **semi-implicit**. For the shallow-water equations, if we treat gravity waves using backward Euler and advection and Coriolis with forward Euler, mark which terms are evaluated at time-level n and which at time-level $n+1$.

$$\frac{\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}}{\Delta t} + \mathbf{u}^{(n\ \ )} \cdot \nabla \mathbf{u}^{(n\ \ )} = -2\Omega \times \mathbf{u}^{(n\ \ )} - g\nabla h^{(n+1)} \tag{3.1}$$

$$\frac{h^{(n+1)} - h^{(n)}}{\Delta t} + \mathbf{u}^{(n\ \ )} \cdot \nabla h^{(n\ \ )} = -h^{(n\ \ )} \nabla \cdot \mathbf{u}^{(n+1)} \tag{3.2}$$

This is solved by rearranging equation 3.1 for $u^{(n+1)}$ and substituting these into equation 3.2. This leads to a semi-discretised version of the wave equation (2.5) or Helmholtz equation which can be solved for $h^{(n+1)}$.

$$\mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} - \Delta t \left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{u}^{(n)} + 2\Omega \times \mathbf{u}^{(n)} + g\nabla h^{(n+1)} \right)$$

$$\Longrightarrow \frac{h^{(n+1)} - h^{(n)}}{\Delta t} + \mathbf{u}^{(n)} \cdot \nabla h^{(n)}$$

$$= -h^{(n)} \nabla \cdot \left(\mathbf{u}^{(n)} - \Delta t \left(\mathbf{u}^{(n)} \cdot \nabla \mathbf{u}^{(n)} + 2\Omega \times \mathbf{u}^{(n)} + g\nabla h^{(n+1)} \right) \right)$$

The terms that make this into a Helmholtz equation are:

$$\frac{h^{(n+1)} - h^{(n)}}{\Delta t} = h^{(n)} \nabla \cdot \left(\Delta t g \nabla h^{(n+1)} \right) + \ldots = \Delta t g h^{(n)} \nabla^2 h^{(n+1)} + \ldots$$

which must be solved implicitly since $h^{(n+1)}$ appears on the RHS and LHS.

A similar procedure is used to solve the Navier-Stokes equations in semi-implicit models of the global atmosphere.

Chapter 4: Conservation

- What should be conserved? [Material from Thuburn, 2008, Ringler et al., 2010]
- How can conservation be achieved?

4.1 Should Weather and Climate Models Conserve Mass?

- ECMWF and the Met Office are the most skillful weather forecasting models and they do *not* conserve mass. Why not?
  - They rely on semi-Lagrangian advection to achieve reasonable time-steps on lat-lon grids

  However we are moving away from lat-lon grids because they do not work well on massively parallel computers
- Mass is conserved in the real atmosphere over all time-scales, regardless of adiabatic processes and friction (neglecting relativistic effects)
- Mass errors $\rightarrow$ pressure errors $\rightarrow$ spurious winds
- Failure to conserve mass means that nothing else can be conserved.
- You do not want to lose all the mass in the atmosphere over a long climate simulation

So *YES* weather and climate models should conserve mass

4.2 Should Weather and Climate Models Conserve Energy?

- Energy is difficult to conserve because it consists of many different types of energy which are calculated separately and transfers between them may not be conservative. It consists of:
  - Available and un-available potential energy (the unavailable potential energy is the potential energy that the atmosphere would have if it were in stationary hydrostatic balance)
  - kinetic energy
  - internal energy (calculated from temperature)
  - chemical energy
- Unavailable PE is much larger than all the others and does not undergo a cascade to smaller scales. $\therefore$ should be conserved exactly.
- Available PE has a time-scale of $\approx$20 days in the atmosphere so spurious sources should lead to slower changes than this.
- Kinetic energy cascades to small scales and there will always be unresolved kinetic energy. $\therefore$ models should dissipate rather than conserve KE (which should lead to a rise in temperature).
- Formal energy conservation helps with model stability.

Avoid large spurious sinks and especially spurious *sources* of energy

4.3 Should Weather and Climate Models Conserve Momentum?

- There is a transfer of momentum with the Earth's surface so momentum is not conserved in the atmosphere.
- Momentum is not dissipated – there is no cascade to small scales.
- Momentum is conserved over very long time-scales in the stratosphere.
- Most models do not conserve momentum
- The stratospheric quasi-biennial oscillation is very difficult to simulate. Related?

Don't know. Usually more focus on energy, enstrophy and vorticity in atmospheric modelling.

4.4 Should Weather and Climate Models Conserve Tracer Variance and Potential Enstrophy?

What is potential enstrophy? ... Variance of the potential vorticity:

$$\text{potential enstrophy} = \frac{1}{2}hq^2$$

where

$$\text{potential vorticity, } q = (\eta + 2\Omega) \cdot \mathbf{k}/h \text{ , vorticity, } \eta = \nabla \times \mathbf{u}$$

- Cascade to small scales then dissipation in about 10 days
- Lots of un-resolved tracer variance and enstrophy
- Conservation $\rightarrow$ spectral blocking at the grid-scale
- $\therefore$ need to make sure that tracer variance and enstrophy are destroyed at the grid scale. How ...
  - Scale selective dissipation or
  - Dissipative advection scheme, eg
    - Odd order finite volume with a limiter
    - Even order semi-Lagrangian

    So that leading order error is dissipative rather than dispersive

4.5 Should Weather and Climate Models Conserve Potential Vorticity?

- PV conservation → correct strength of weather systems
- PV is derived from velocity so conservation requires careful numerics

4.6 How can Mass be Conserved?

Air density, ρ, satisfies the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u}\rho) = 0 \tag{4.1}$$

where $\mathbf{u}$ is the wind. This is a conservation equation for ρ. Let us assume that ρ and $\mathbf{u}$ vary only in the x direction and that our periodic domain goes from $x = 0$ to $x = 2\pi$ (for example around the equator). Then the total mass of air is:

$$M = \int_0^{2\pi} \rho \; dx \tag{4.2}$$

If we solve eqn (4.1) using a finite volume scheme we get:

$$\rho_j^{(n+1)} = \rho_j^{(n)} - \Delta t \frac{u_{j+1/2}\rho_{j+1/2} - u_{j-1/2}\rho_{j-1/2}}{\Delta x} \tag{4.3}$$

where $u_{j\pm 1/2}$ and $\rho_{j\pm 1/2}$ can be defined in any way from the values of u_j and ρ_j at the grid points (eg interpolation). The total mass in the computational domain at time n is:

$$M^{(n)} = \sum_{j=1}^{j=N} \Delta x \rho_j^{(n)}. \tag{4.4}$$

To find $M^{(n+1)}$ as predicted from the finite volume scheme, we can substitute eqn (4.3) into (4.4):

$$
\begin{aligned}
M^{(n+1)} &= \sum_{j=1}^{j=N} \Delta x \rho_j^{(n+1)} = \sum_{j=1}^{j=N} \left\{ \Delta x \rho_j^{(n)} - \Delta t \left(u_{j+1/2}\rho_{j+1/2} - u_{j-1/2}\rho_{j-1/2} \right) \right\} && (4.5)\\
&= M^{(n)} - \Delta t \left(\sum_{j=1}^{j=N} u_{j+1/2}\rho_{j+1/2} - \sum_{j=1}^{j=N} u_{j-1/2}\rho_{j-1/2} \right) && (4.6)\\
&= M^{(n)} - \Delta t \left(\sum_{j=1}^{j=N} u_{j+1/2}\rho_{j+1/2} - \sum_{j=0}^{j=N-1} u_{j+1/2}\rho_{j+1/2} \right) && (4.7)\\
&= M^{(n)} - \Delta t \left(u_{N+1/2}\rho_{N+1/2} - u_{1/2}\rho_{1/2} \right) && (4.8)
\end{aligned}
$$

and since we have periodic boundary conditions, $u_{N+1/2}\rho_{N+1/2} = u_{1/2}\rho_{1/2}$ proving that $M^{(n+1)} = M^{(n)}$ and hence mass is conserved.

If instead we discretised the advective form of the continuity equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \tag{4.9}$$

using finite differences or semi-Lagrangian, conservation would not be so easy to achieve.

4.7 How can Energy be Conserved?

One possible technique would be to solve a conservation equation for total energy and then calculate temperature from the energy. However this is not done in atmospheric models. Why not? ... Instead we can consider how energy can be conserved in the vector invariant form of the linearised shallow-water equations, solving for $\mathbf{u}$ and h:

$$\frac{\partial \mathbf{u}}{\partial t} + \eta \times \mathbf{u} = -g\nabla h - \nabla K \tag{4.10}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0 \tag{4.11}$$

where $\eta = \nabla \times \mathbf{u} + 2\Omega$ is the total vorticity and $K = \frac{1}{2}|\mathbf{u}|^2$ is the kinetic energy. The energy is defined as:

$$E = \frac{1}{2}gh^2 + hK. \tag{4.12}$$

If we calculate $h\mathbf{u}\cdot$(4.10)+$(gh+K)$ (4.11) we can derive the conservation equation for E:

$$h\mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} + h\mathbf{u} \cdot (\eta \times \mathbf{u}) + (gh+K)\frac{\partial h}{\partial t} + (gh+K)\nabla \cdot (h\mathbf{u}) = -gh\mathbf{u} \cdot \nabla h - h\mathbf{u} \cdot \nabla K \tag{4.13}$$

Using the vector calculus identities:

$$\begin{aligned} \mathbf{u}\cdot(\eta\times\mathbf{u}) &= 0 \\ K\nabla\cdot(h\mathbf{u}) + h\mathbf{u}\cdot\nabla K &= \nabla\cdot(Kh\mathbf{u}) \end{aligned}$$

eqn (4.13) can be rearranged to give:

$$\frac{\partial E}{\partial t} + \nabla\cdot(h^2\mathbf{u} + hK\mathbf{u}) = 0. \tag{4.14}$$

The term $\nabla\cdot(h^2\mathbf{u}+hK\mathbf{u})$ is the divergence of a flux so it cannot create or destroy energy, just move energy around. Therefore, for a SWE model to conserve energy, it must have discrete versions of the above vector calculus identities. These are called *mimetic* properties. The mimic properties of the real system.

4.8 How can Potential Vorticity be Conserved?

The PV equation can be derived by taking the curl of the momentum equation (in vector invariant form):

$$\nabla\times\frac{\partial\mathbf{u}}{\partial t} + \nabla\times(\eta\times\mathbf{u}) = \cancelto{0}{-\nabla\times(g\nabla h + \nabla K)} \tag{4.15}$$

Considering the 2D flow of the SWE and uing the vector calculus identities:

$$\nabla\times\nabla h = 0$$

this gives:

$$\frac{\partial hq}{\partial t} + \nabla\cdot(qh\mathbf{u}) = 0$$

which is a conservation equation for the PV, $q = (\eta + 2\Omega)\cdot\mathbf{k}/h$. So for a model to conserve PV, it must have a discrete equivalent of curl free gradients. This is another *mimetic* property.