Fourier Analysis

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This is brief introduction to Fourier analysis and how it is used in atmospheric and oceanic science, for:

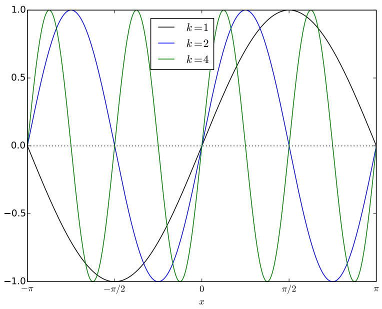
- Analysing data (eg climate data)
- Numerical methods
- Numerical analysis of methods

1 Fourier Series

Any periodic, integrable function, $f(x)$ (defined on $[-\pi,\pi]$), can be expressed as a Fourier series; an infinite sum of sines and cosines:

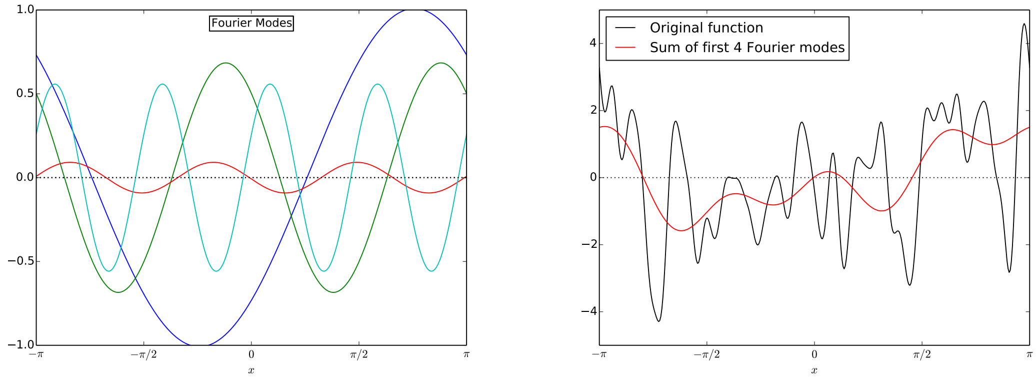
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx \quad (1)$$

- The a_k and b_k are the Fourier coefficients.
- The sines and cosines are the Fourier modes.
- k is the wavenumber - number of complete waves that fit in the interval $[-\pi,\pi]$
  $\sin kx$ for different values of k



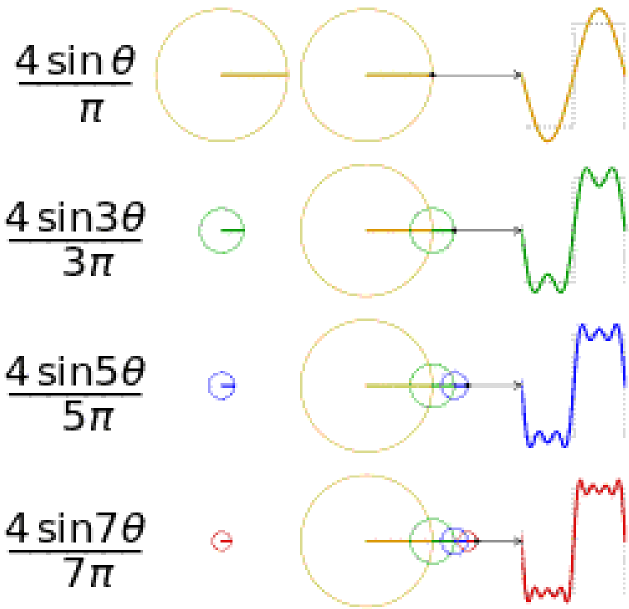
- The wavelength is $\lambda =$
- The more Fourier modes that are included, the closer their sum will get to the function.

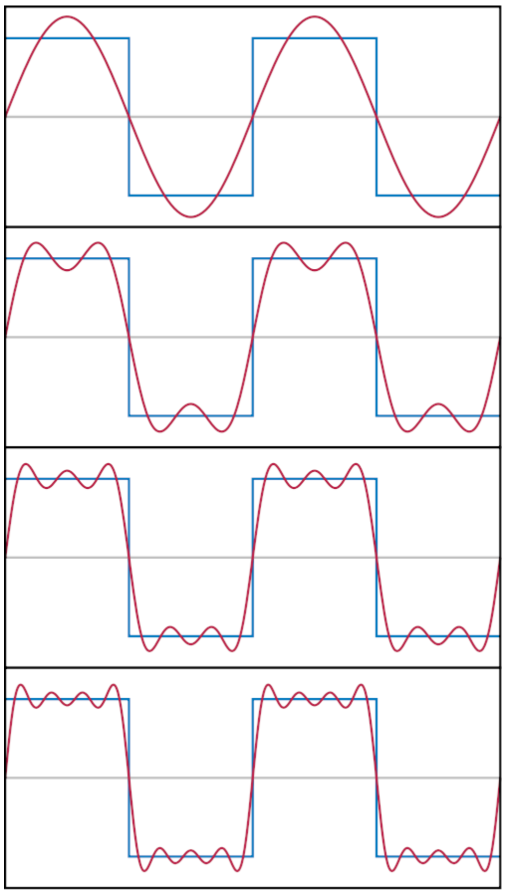
Sum of First 4 Fourier Modes of a Periodic Function



The first four Fourier modes of a square wave.
The additional oscillations are "*spectral ringing*"

Each mode can be represented by motion around a circle. ↓
The motion around each circle has a speed and a radius. These represent the wavenumber and the Fourier coefficients. Which is which?





Equivalently, equation (1) can be expressed as an infinite sum of exponentials using the relation $e^{i\theta} = \cos\theta + i\sin\theta$ where $i = \sqrt{-1}$:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx = \sum_{k=-\infty}^{\infty} A_k e^{ikx}. \qquad (2)$$

Exercise

Evaluate the A_ks in terms of the a_ks and b_ks.

2 Fourier Transform

- The Fourier Transform transforms a function f which is defined over space (or time) into the frequency domain, so that it is defined in terms of Fourier coefficients.
- The Fourier transform calculates the Fourier coefficients as:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx\ ,\ \ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

3 Discrete Fourier Transform

A discrete Fourier Transform converts a list of $2N+1$ equally spaced samples of a real valued, periodic function, f_n, to the list of the first $2N+1$ complex valued Fourier coefficients:

$$A_k = \frac{1}{N}\sum_{n=-N}^{N} f_n\, e^{-i\pi nkx/N}.$$

The truncated Fourier series:

$$f(x) \approx \sum_{k=-N}^{N} A_k e^{ikx}$$

is an approximation to the function f which fits the sampled points, f_n, exactly.

On a computer this is done with a **Fast Fourier Transform** (or `fft`). The inverse Fourier transform (sometimes called `ifft`) transforms the Fourier coefficients back to the f values (transforming from spectral back to real space):

$$\begin{array}{ccc} f_0, f_1, f_2, \cdots f_{2N} & \xrightarrow{\texttt{fft}} & A_0, A_1, \cdots A_{2N} \\ A_0, A_1, \cdots A_{2N} & \xrightarrow{\texttt{ifft}} & f_0, f_1, f_2, \cdots f_{2N} \end{array}$$

4 Differentiation and Interpolation

If we know the Fourier coefficients, A_k, of a function f then we can calculate the gradient of f at any point, x: If

$$f(x) = \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=0}^{\infty} b_k \sin kx = \sum_{k=-\infty}^{\infty} A_k e^{ikx} \tag{3}$$

then

$$f'(x) = \tag{4}$$

and the second derivative:

$$f''(x) = \tag{5}$$

These have spectral accuracy; the order of accuracy is as high as the number of points. Similarly equation 1 or 2 can be used directly to interpolate f onto an undefined point, x. Again, the order of accuracy is spectral.

5 Spectral Models

- ECMWF use a spectral model.
- The prognostic variables are transformed between physical and spectral space using `ffts` and `iffts`.
- Gradients are calculated very accurately in spectral space

6 Wave Power and Frequency

- If a function, f, has Fourier coefficients, a_k and b_k, then wavenumber k has power $a_k^2 + b_k^2$.
- A plot of wave frequency versus power is referred to as the power spectrum. Before we learn how power spectra are used, we will have some revision questions...

7 Recap Questions

1. In the Fourier decomposition

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

   what are:

   (a) the Fourier coefficients

   (b) the Fourier modes

   (c) the wavenumbers (or frequencies)

   (d) the power of a given wavenumber

2. How would you describe the operation:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \;, \;\; b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

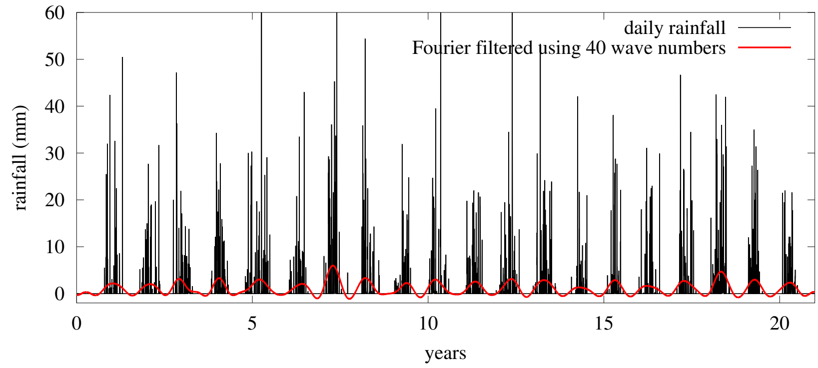
3. Given a list of $2N+1$ equally spaced samples of a real valued, periodic function, f_n, how would you describe the following operation to convert this into a a list of $N+1$ values:

$$A_k = \frac{1}{N} \sum_{n=-N}^{N} f_n \, e^{-i\pi knx/N}$$

4. What is the wavelength of a wave described by $\sin 4x$

8 Analysing Power Spectra

Daily rainfall at a station in the Middle East for 21 years



Obervations about the Truncated Fourier filtered rainfall:

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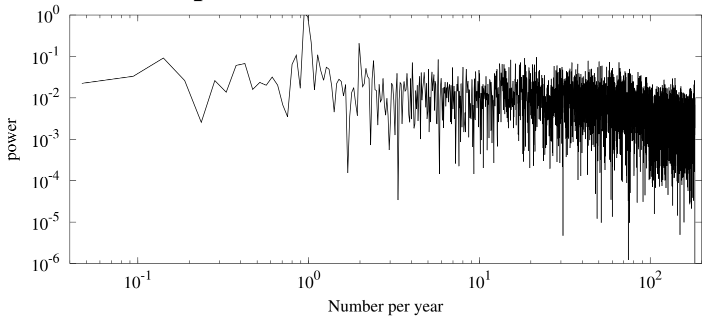
Power Spectrum of Middle East Rainfall



Observations

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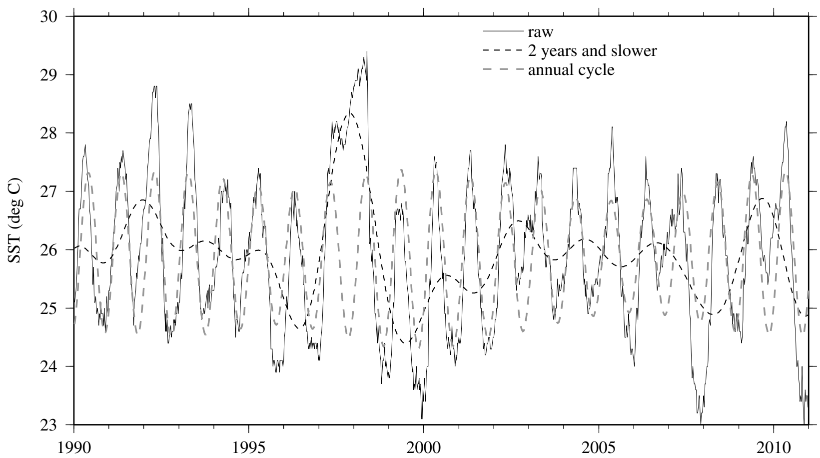
number per year = wavenumber$\times$365/total number of days

Time Series of the Nino 3 sea surface temperature (SST)

The SST in the Nino 3 region of the equatorial Pacific is a diagnostic of El Nino



How were the dashed lines generated?

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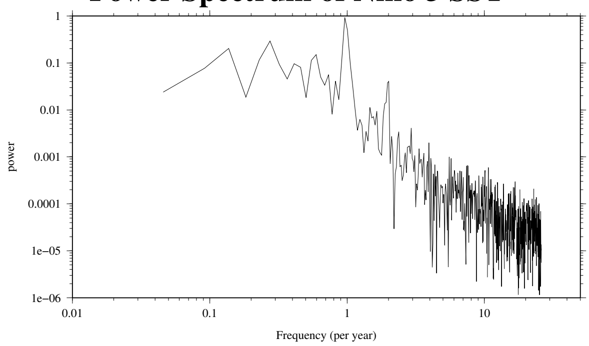
Power Spectrum of Nino 3 SST



Observations

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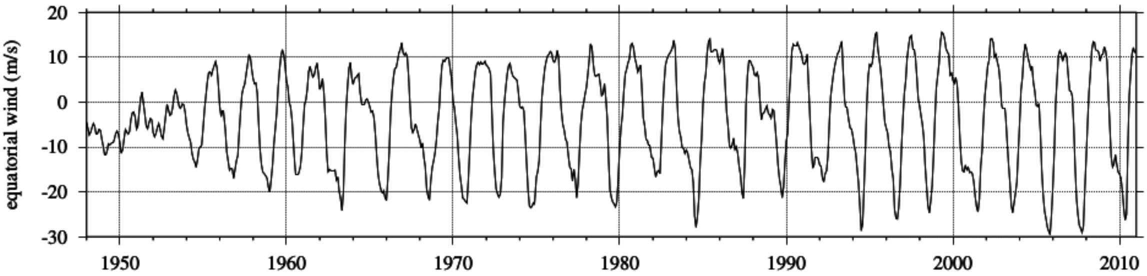
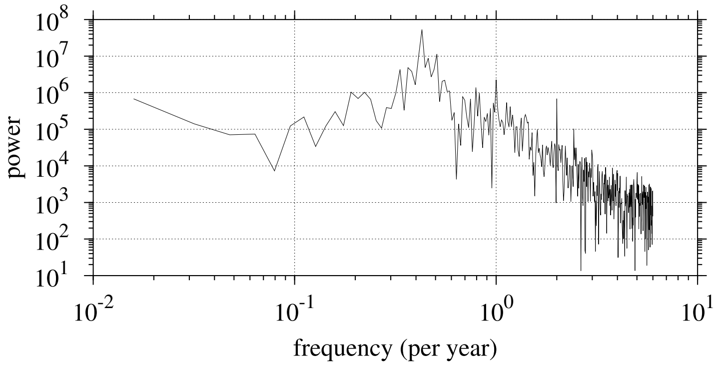
Time Series of the Quasi-Biennial Oscillation (QBO)

The QBO is an oscillation of the equatorial zonal wind between easterlies and westerlies in the tropical stratosphere which has a mean period of 28 to 29 months:



Power Spectrum of QBO



Observations

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