Wave propagation on adapting, inhomogeneous, and unstructured grids

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Outline

- Why wave propagation?
- Benefits of staggered grids
- Effects of grid non-uniformity
- Effects of the grid structure; spurious modes
- Generation of imbalance under adaptation

Why wave propagation?

Large-scale atmosphere and ocean flows are energetically dominated by slow, **balanced**, Rossby modes.

Fast acoustic and inertio-gravity modes are energetically weak, but are responsible for **adjustment** towards balance.

There is currently no suitable **acoustically filtered** equation set.

"Better learn balance. Balance is key. Balance good... Everything good."
(The Karate Kid, 1984)

Simplest example: 1D gravity waves

Linearized, 1D, non-rotating, shallow water equations

$$\frac{\partial \Phi}{\partial t} + \Phi_0 \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + \frac{\partial \Phi}{\partial x} = 0$$

Look for wavelike solutions

$$\Phi = Re\left\{\hat{\Phi}\exp[\mathrm{i}(kx - \omega t)]\right\} \qquad u = Re\left\{\hat{u}\exp[\mathrm{i}(kx - \omega t)]\right\}$$

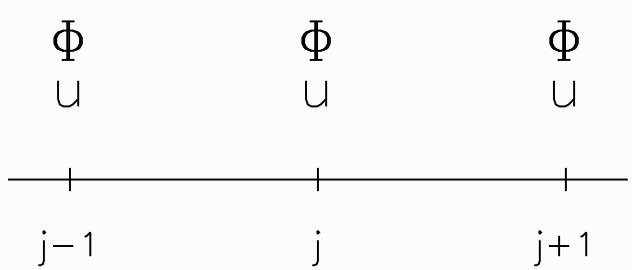
to find the dispersion relation

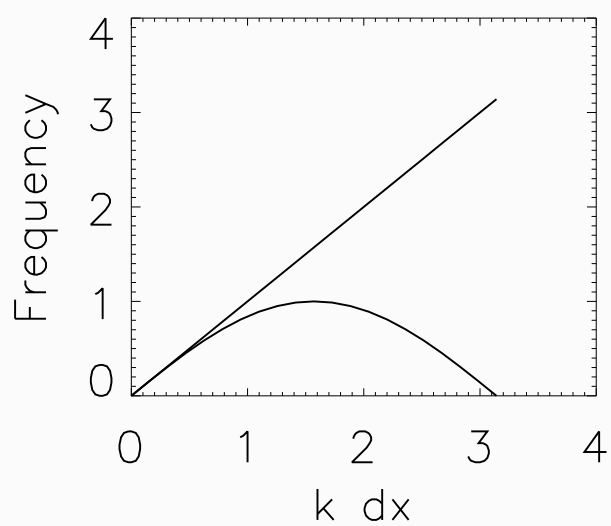
$$\omega^2 = k^2 \Phi_0$$

On an unstaggered grid

$$\frac{\partial u_j}{\partial t} + \frac{\Phi_{j+1} - \Phi_{j-1}}{2\Delta x} = 0 \qquad \frac{\partial \Phi_j}{\partial t} + \frac{u_{j+1} - u_{j-1}}{2\Delta x} = 0$$

$$k \to \sin(k\Delta x)/\Delta x$$

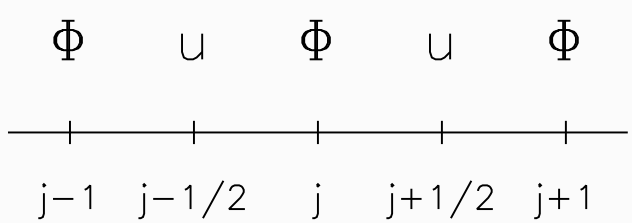


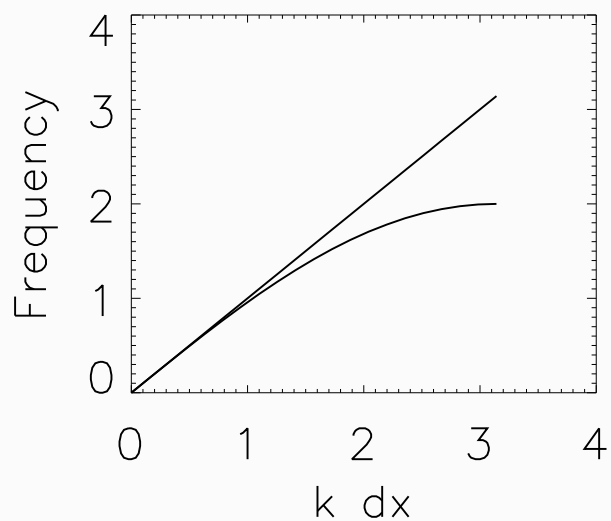


On a staggered grid

$$\frac{\partial u_{j+1/2}}{\partial t} + \frac{\Phi_{j+1} - \Phi_j}{\Delta x} = 0 \qquad \frac{\partial \Phi_j}{\partial t} + \frac{u_{j+1/2} - u_{j-1/2}}{\Delta x} = 0$$

$$k \to \sin(k\Delta x/2)/(\Delta x/2)$$





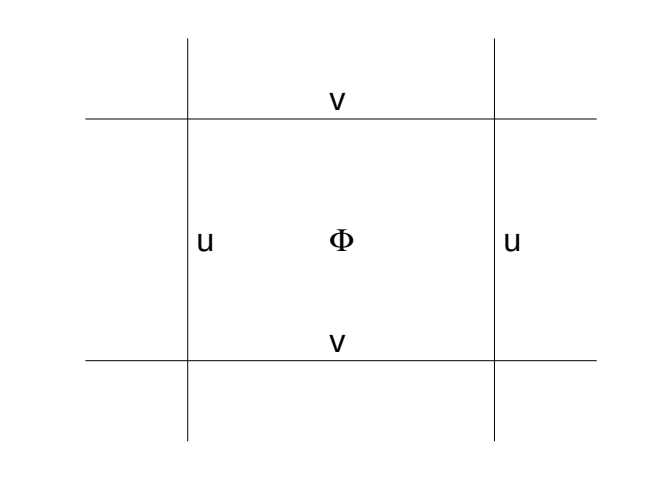
Matlab demo of unstaggered and staggered grids

Matlab demo of parasitic mode

In 2D suitable grids are

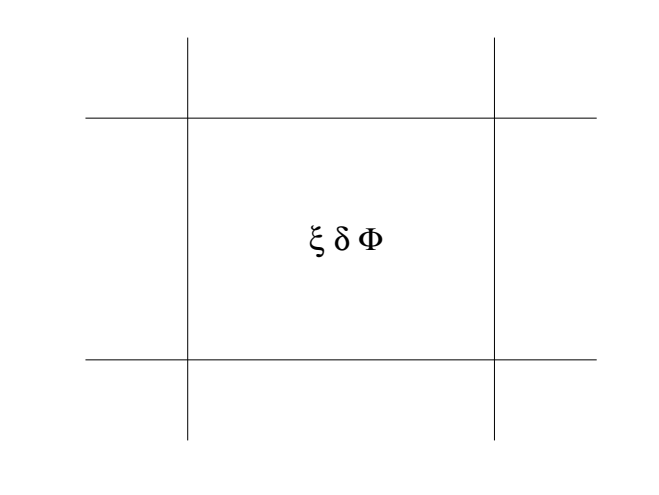
C-grid

Provided the Rossby radius is well-resolved;



Z-grid

But need to solve elliptic problems at each step.



There are issues related to

- Grid non-uniformity
- Grid structure
- The adaptation process

Effects of grid non-uniformity

- Waves resolved in one region but not in another must be reflected or dissipated
- Numerical dispersion relation is spatially dependent, leading to spurious refraction and scattering

Non-uniform 1D grid

Centred differences, advection (Vichnevetsky):

- Can calculate **reflection coefficient** for an abrupt jump in resolution
- **Asymptotically no reflection**

Matlab demo

Analogous results hold for linearized SWE ($f = 0$) on unstaggered and staggered grids

2D, 3D?

Expect refraction (refractive index?), scattering, ...

Effects of grid structure

Think 2D shallow water equations.

On **quadrilateral C-grids**, if we have N cells then we have N degrees of freedom each in u, v and Φ...

... resulting in $2N$ inertio-gravity modes and N Rossby modes.

What happens on **other grids structures**?

We understand **equilateral triangles**:

N dof in Φ, $3N/2$ in velocity ($N/2$ in vorticity, N in divergence)...

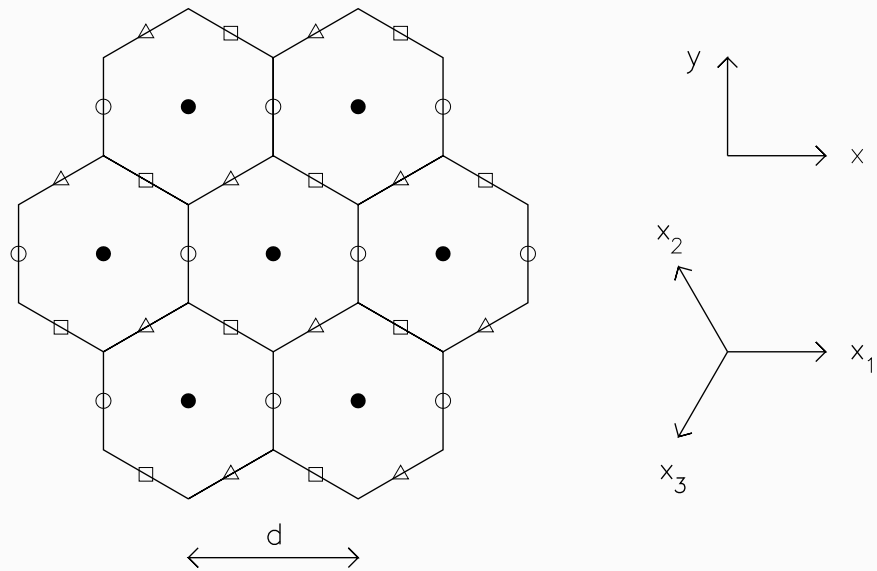
...resulting in $2N$ **'inertio-gravity modes'** and $N/2$ **Rossby modes**.

and **hexagons**:

N dof in Φ, $3N$ in velocity ($2N$ in vorticity, N in divergence)...

...resulting in $2N$ **inertio-gravity modes** and $2N$ **'Rossby' modes**.

Hexagonal C-grid



Quartic f-plane dispersion relation

$$\omega^4 - \omega^2 \left\{ \frac{f_0^2}{3} \left(a_1^2 + a_2^2 + a_3^2\right) + \frac{8\Phi_0}{3d^2} \left(s_1^2 + s_2^2 + s_3^2\right) \right\} = 0.$$

where $c_j = \cos\left(k_j d/2\right),\ s_j = \sin\left(k_j d/2\right),\ k_j = \mathbf{k}.\hat{\mathbf{x}}_j,\ a_1 = (2c_2c_3 + c_1)/3$ etc.

The discretization of the Coriolis terms is crucial

On an f-plane, **geostrophic modes should be stationary**.

Require a discrete analogue of $\partial_t \xi + f\delta = 0$

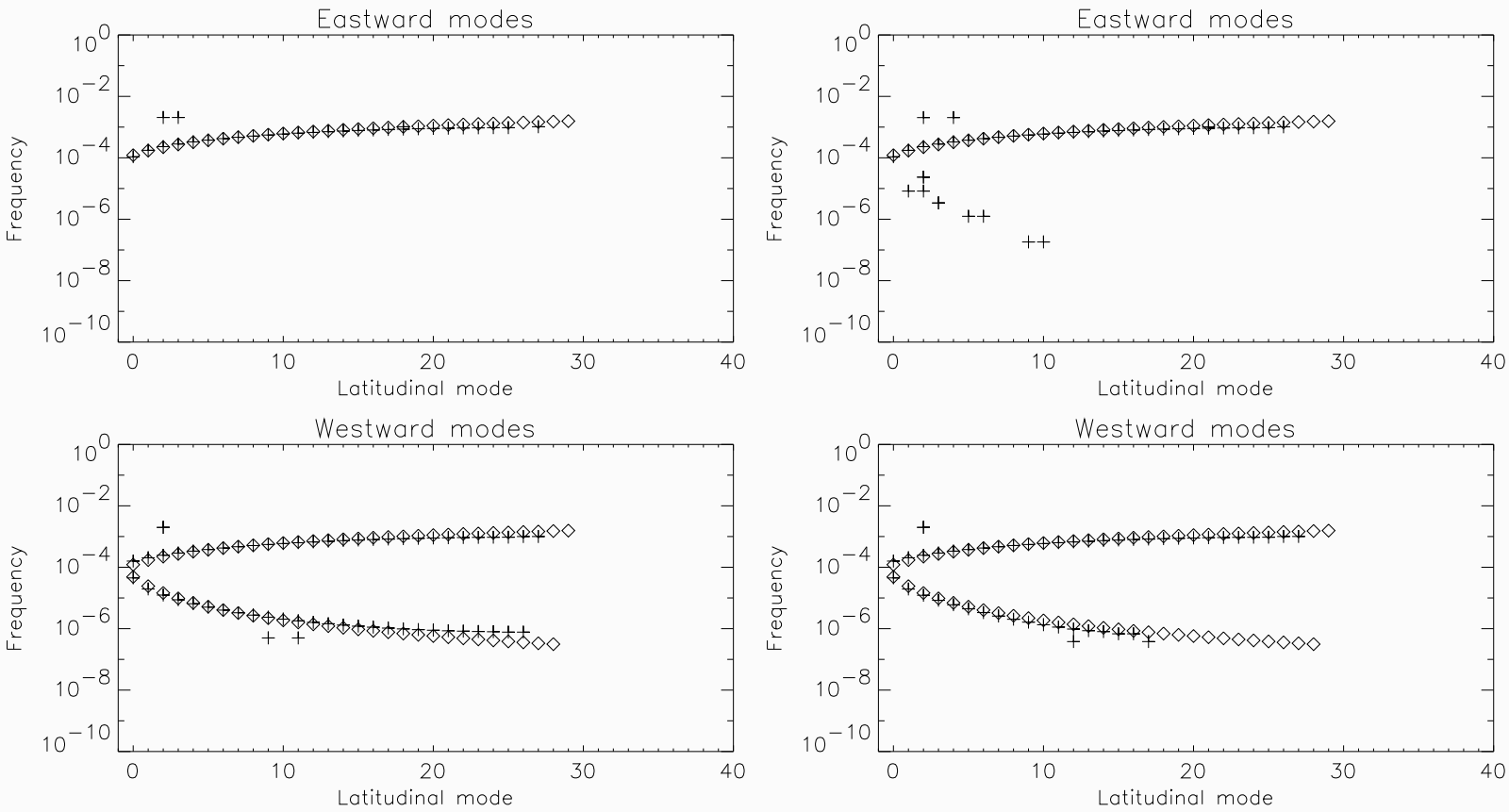
Write $u_e^\perp = \sum_{e'} w_e^{e'} u_{e'}$

We now know how to construct the weights $w_e^{e'}$ to get stationary geostrophic modes and keep energy conservation on **arbitrary C-grids**.

Stationary geostrophic modes for constant f is a **pre-requisite for good Rossby mode behaviour** when f varies.

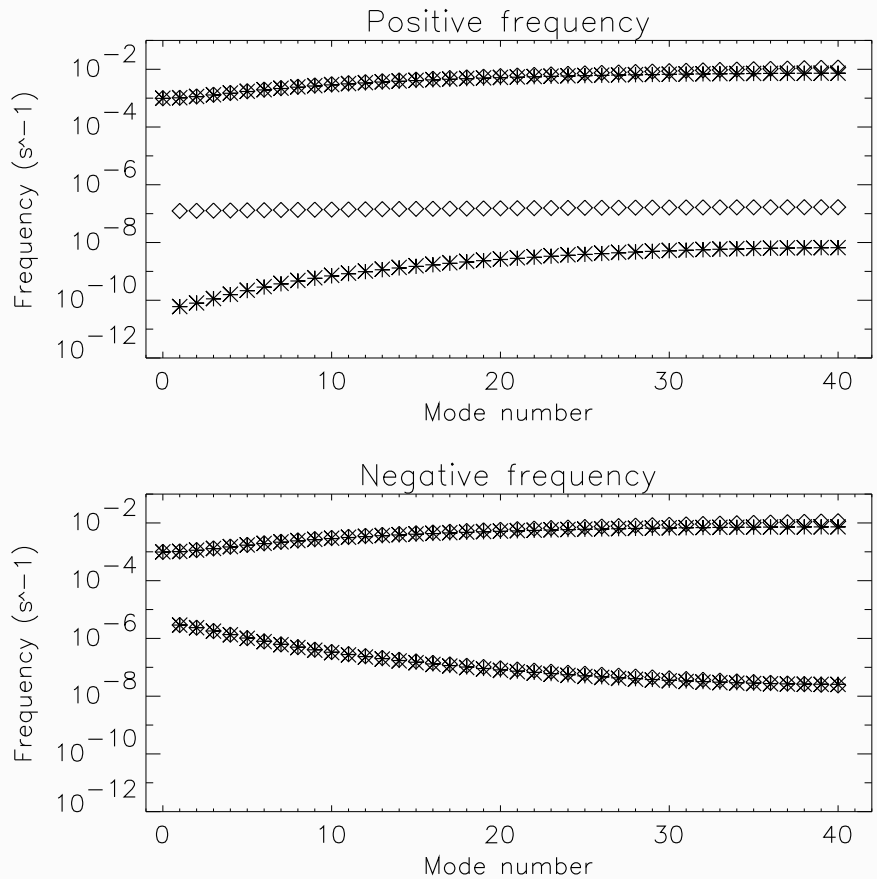
$$\frac{\partial u}{\partial t} - \overline{\frac{f}{\cos\phi}\overline{v\cos\phi}^{\phi}}^{\lambda} + \frac{1}{a\cos\phi}\delta_\lambda\Phi = 0$$

$$\frac{\partial v}{\partial t} + \overline{f\overline{u}^{\lambda}}^{\phi} + \frac{1}{a}\delta_\phi\Phi = 0$$


Eastward modes
Eastward modes
Westward modes
Westward modes
Frequency
Latitudinal mode
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
0
10
20
30
40

Back on the hexagonal C-grid...

The extra Rossby mode branch is unphysical



Geostrophic velocity

On a triangular C-grid the **geostrophic $\mathbf{v}$ does not exist** for arbitrary Φ - not enough velocity degrees of freedom.

On a hexagonal C-grid the **geostrophic $\mathbf{v}$ is non-unique** - too many velocity degrees of freedom.

Other grid structures?

It seems highly likely that unstructured and locally refined C-grids will support spurious numerical wave modes, and that the geostrophic $\mathbf{v}$ will be either non-existent, non-unique, or both.

Generation of waves by the adaptation process

Due to imperfect **compatibility** between the numerical representation of balance and the remeshing process. Possibility of runaway adaptation.

Matlab demo

Note that more accurate remeshing, on its own, does not solve the problem.

Spurious imbalance depends on $\Delta x/a$ as well as $\Delta x/\lambda$.

Balance is a **non-local** property: an accurate treatment is likely to require the solution of elliptic problems.

Some suggestions for future work

Grid inhomogeneity:

- Need to build in some dissipation - but what is optimal?
- Can ideas from non-reflecting boundary conditions help?

C-grid-related spurious wave modes

- What kinds of wave modes exist on an unstructured C-grid?
- What about the Z-grid?

Imbalance under adaptation

- Can it be minimized with a local scheme?
- Balance vs conservation?
- What about the Z-grid?