Explicit Two-Step Peer Methods for the Compressible Euler Equations

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Introduction

In atmospheric models the highest-frequency modes are often not the physical modes of interest. On the other hand severe stability constrains for the numerical integrator arise from those meteorologically irrelevant modes. A common strategy to avoid this problem is a splitting approach: The differential equation is split into two parts. The slow part is integrated with one numerical method and a time step size restricted by the CFL number of the low-frequency modes. For the integration of the high-frequency modes a simpler method is used together with smaller time steps so that these small time step sizes satisfy the CFL condition dictated by the high-frequency modes.

The disadvantage of the commonly used split-explicit methods is the fact that the high-frequency modes still constrain the maximal time step size if no additional damping term such as divergence damping is used. For split-explicit Runge-Kutta methods there is the constraint $c_s \Delta t/\Delta x < \pi$ where c_s is the speed of sound. In contrast the CFL number of advection e.g. for RK3 is $\sqrt{3}$ which means that in case of maximal wind speeds below of 190ms^{-1} the acoustic modes constrain the maximal time step size.

We present a new methodology to describe time-splitting methods. The basic principle of the presented approach is the assumption that one part, the fast part, of the split differential equation can be solved analytically so that stability and order investigations can be made for the underlying method which solves the slow part. With this methodology we consider common split-explicit Runge-Kutta methods like RK3 of Wicker and Skamarock, a new class of generalized split-explicit 'Runge-Kutta' methods developed by Wensch and Knoth and we present a new class of split-explicit methods which use peer methods as underlying method for the solution of split differential equations.

Time-Splitting Methods

We consider the numerical solution of autonomous split differential equations

$$\dot{y} = f(y) + g(y), \qquad y(t_0) = y_0 \in \mathbb{R}^n \tag{1}$$

in $[t_0, t_e]$ where f represents the slow part and g the fast part.

We use a more technical way to introduce split-explicit Runge-Kutta methods than Wicker and Skamarock did because we do not restrict us to one integrator for the fast part of (1), our formulation of the split-explicit scheme contains the fast part as initial value problem. With this view we can focus on the properties of the main solver for the slow part, i.e. on the underlying Runge-Kutta and peer methods. Furthermore this approach will simplify the derivation of order conditions. The widely used split-explicit Runge-Kutta method RK3 by Wicker and Skamarock 2002 is then given by

$$Z_{ni}(0) = y_n$$

$$\frac{\partial}{\partial \tau} Z_{ni}(\tau) = \frac{1}{c_i} \sum_{j=1}^{i-1} a_{ij} f(Y_{nj}) + g(Z_{ni}(\tau))$$

$$Y_{ni} = Z_{ni}(c_i h), \qquad y_{n+1} = Y_{n,s+1}$$

where the coefficients of the underlying Runge-Kutta method are defined by the Butcher tableau:

$$\begin{array}{c|ccc} 0 & & & \\ 1/3 & 1/3 & & \\ 1/2 & 0 & 1/2 & \\ \hline & 0 & 0 & 1 \end{array}$$

Wensch et al. (2009) generalized split-explicit Runge-Kutta methods by the inclusion of fixed tendencies of previous stages and by starting the integration of the fast part at some intermediate point instead of the beginning of the time intervals. Their scheme WKG reads:

$$Z_{ni}(0) = y_n + \sum_{j=1}^{i-1} \alpha_{ij}(Y_{nj} - y_n)$$

$$\frac{\partial}{\partial \tau} Z_{ni}(\tau) = \frac{1}{\alpha_i}\left(\frac{1}{h} \sum_{j=1}^{i-1} \gamma_{ij}(Y_{nj} - y_n) + \sum_{j=1}^{i-1} \beta_{ij} f(Y_{nj}) \right) + g(Z_{ni}(\tau))$$

$$Y_{ni} = Z_{ni}(\alpha_i h), \qquad y_{n+1} = Y_{n,s+1}$$

In practice the integration of the fast differential equation is done with forward-backward Euler.

A further more generalization of time-splitting schemes is to use peer methods for the integration of the slow part. Peer methods are general linear methods with the same order in every stage. Jebens et al. (2009) used the representation

$$Z_{ni}(0) = \sum_{j=1}^{s} b_{ij} Y_{n-1,j} + \sum_{j=1}^{i-1} s_{ij} Y_{nj}$$

$$\frac{\partial}{\partial \tau} Z_{ni}(\tau) = \frac{1}{\alpha_i}\left(\sum_{j=1}^{s} a_{ij} f(Y_{n-1,j}) + \sum_{j=1}^{i-1} r_{ij} f(Y_{nj}) \right) + g(Z_{ni}(\tau))$$

$$Y_{ni} = Z_{ni}(\alpha_i h).$$

Linear Stability Analysis

For the linear stability analysis of split-explicit methods we consider the linear system of equations

$$u_t = -c_s p_x - U u_x$$
$$p_t = -c_s u_x - U p_x$$

with $c_s >> U$. Using central differences for the fast part and the third-order upwind scheme for the advection part together with the von Neumann stability ansatz leads to the simplified ODE

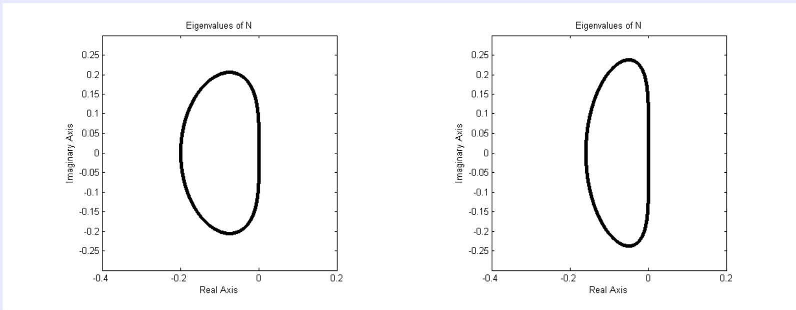
$$\dot{y} = Ly + Ny$$

with

$$L = -\frac{c_s}{\Delta x} 2i \sin(\pi \Delta x/k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

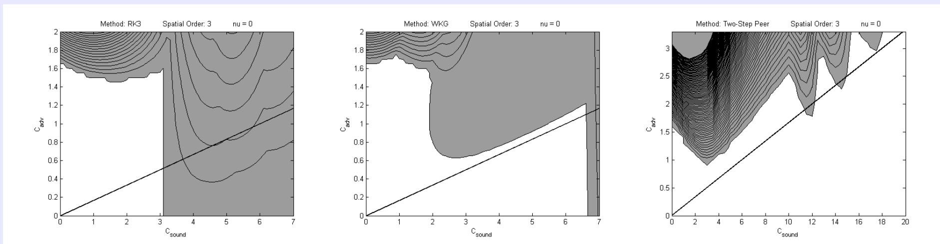
$$N = -\frac{U}{6\Delta x}\Big(\cos(4\pi\Delta x/k) - 4\cos(2\pi\Delta x/k) + 3 + i(-\sin(4\pi\Delta x/k) + 8\sin(2\pi\Delta x/k)) \Big) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where L represents the fast part and N the slow modes. The eigenvalues of the fast part are purely imaginary while the eigenvalues of the slow part have small real parts:



Eigenvalues of the slow part for third-order upwind (left) and fifth-order upwind (right).

Applying the 3 introduced methods to this test equation results in the following stability diagrams, grey colour denotes unstable regions, the variables are the Courant numbers for advection $U\frac{\Delta t}{\Delta x}$ (vertical axis) and sound $c_s\frac{\Delta t}{\Delta x}$ (horizontal axis):



Stability regions of RK3 (left), WKG (middle) and Peer (right) for the linear test equation.

Application to the Compressible Euler Equations

We applied the split-explicit methods to the 2D compressible Euler equations in flux form. We use a finite volume spatial discretization on an Arakawa C grid, so the winds are defined on the cell edges while all scalar variables are defined at the cell centers. We implemented the model in MATLAB with block-static adaptive grids and cut cells for the representation of orography. The fast modes can be integrated with forward-backward Euler, Störmer-Verlet and with the trapezoidal rule. The equations are:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial z}$$

$$\frac{\partial \rho u}{\partial t} = -\frac{\partial \rho uu}{\partial x} - \frac{\partial \rho wu}{\partial z} - \frac{\partial p}{\partial x}$$

$$\frac{\partial \rho w}{\partial t} = -\frac{\partial \rho uw}{\partial x} - \frac{\partial \rho ww}{\partial z} - \frac{\partial p}{\partial z} - \rho g$$

$$\frac{\partial \rho\theta}{\partial t} = -\frac{\partial \rho u\theta}{\partial x} - \frac{\partial \rho w\theta}{\partial z}$$

$$p = \left(\frac{R\rho\theta}{p_0^{\kappa}} \right)^{\frac{1}{1-k}}$$

Here u and w are the horizontal and vertical winds, ρ is the density, p the pressure, θ the potential temperature and g the acceleration of gravity. The prognostic variables are ρ, ρu, ρw and $\rho\theta$, the pressure p is given diagnostically by the equation of state.

Overview

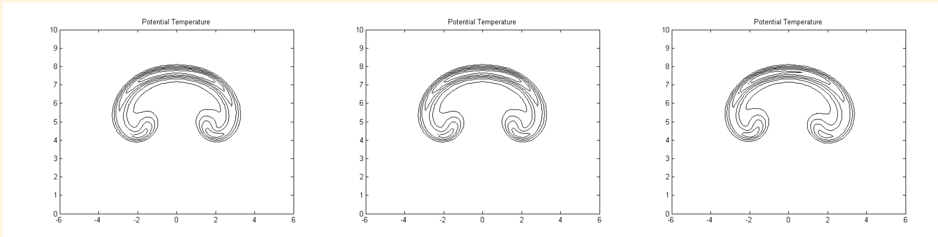
The presented methods have the following properties where p denotes the order, s the stage number, $||\alpha||_1$ is the sum of the fast integration intervals, C_{adv} is the maximal Courant number with respect to advection and C_{sound} is the maximal Courant number for sound waves.

Method	p	s	$\|\|\alpha\|\|_1$	C_{adv}	C_{sound}
RK3	2	3	1.83	1.7	3
WKG	3	3	1.93	1.7	6.5
Peer	2	3	1.44	1.7	>20

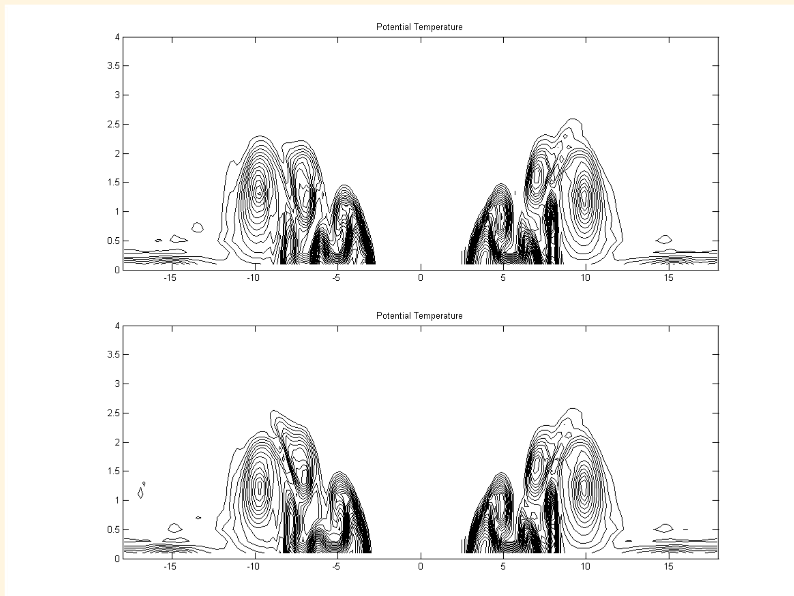
Results

We apply the methods to 2 standard tests: For the rising bubble the initial atmosphere is adiabatically stratified with the exception of a perturbation of the potential temperature which is up to 2 degrees warmer than the surrounding air and causes the bubble to rise. For the density current test the bubble is up to 15 degrees colder than the surrounding air so that it will sink and after crashing the ground several eddies will form. To make the tests more stringent a horizontal background wind (from the left) of 20ms^{-1} is added.

For the rising bubble test the horizontal resolution is 125m and the third-order upwind scheme is used while for the density current test the resolution is 200m and the fifth-order upwind scheme is used. The maximal (horizontal) wind speeds that occur are 28ms^{-1} for the first test respectively 50ms^{-1} for the second test. Because the maximal CFL numbers of the advection schemes are 1.7 (third-order) respectively 1.4 (fifth-order) the time step restrictions from advection are $\Delta t \leq 7.5s$ (rising bubble) respectively $\Delta t \leq 5.6s$ (density current).



Potential temperature after 1000s for the rising bubble test computed with RK3 with $\Delta t = 0.75s$ (left), WKG with $\Delta t = 1.5s$ (middle) and Peer with $\Delta t = 7s$ (right).



Potential temperature after 900s for the density current test computed with RK3 with $\Delta t = 1s$ (top) and Peer with $\Delta t = 5s$ (bottom).

Bibliography

S. Jebens, O. Knoth and R. Weiner, Explicit Two-Step Peer Methods for the Compressible Euler Equations, *Monthly Weather Review* (2009), doi:10.1175/2008MWR2671.1.

J. Wensch, O. Knoth and A. Galant, Multirate infinitesimal step methods for atmospheric flow simulation, submitted to *BIT Numerical Mathematics*.

L.J. Wicker and W.C. Skamarock, Time-Splitting Methods for Elastic Models Using Forward Time Schemes, *Monthly Weather Review* (2002), Volume 130, pp. 2088-2097.