# VORTICITY AND DIVERGENCE EQUATIONS

Ross Bannister, Data Assimilation Research Centre, University of Reading

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## The horizontal momentum equations

$$\frac{\partial u}{\partial t} + (\vec{v} \cdot \nabla) u - fv = -\alpha \frac{\partial p}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + (\vec{v} \cdot \nabla) v + fu = -\alpha \frac{\partial p}{\partial y} \quad (2)$$

$$\alpha = \frac{1}{\rho}, \ \nabla = \begin{pmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{pmatrix}, \ \vec{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

## Differentiate each equation with respect to x and y

$$\frac{\partial}{\partial x}(1) \Rightarrow \quad \frac{\partial}{\partial t}\frac{\partial u}{\partial x} + (\vec{v} \cdot \nabla)\frac{\partial u}{\partial x} + (\frac{\partial \vec{v}}{\partial x} \cdot \nabla) u - f\frac{\partial v}{\partial x} = -\alpha \frac{\partial^2 p}{\partial x^2} - \frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial x} \quad (3)$$

$$\frac{\partial}{\partial x}(2) \Rightarrow \quad \frac{\partial}{\partial t}\frac{\partial v}{\partial x} + (\vec{v} \cdot \nabla)\frac{\partial v}{\partial x} + (\frac{\partial \vec{v}}{\partial x} \cdot \nabla) v + f\frac{\partial u}{\partial x} = -\alpha \frac{\partial^2 p}{\partial x \partial y} - \frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} \quad (4)$$

$$\frac{\partial}{\partial y}(1) \Rightarrow \quad \frac{\partial}{\partial t}\frac{\partial u}{\partial y} + (\vec{v} \cdot \nabla)\frac{\partial u}{\partial y} + (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) u - f\frac{\partial v}{\partial y} - \frac{\mathrm{d}f}{\mathrm{d}y}v = -\alpha \frac{\partial^2 p}{\partial x \partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x} \quad (5)$$

$$\frac{\partial}{\partial y}(2) \Rightarrow \quad \frac{\partial}{\partial t}\frac{\partial v}{\partial y} + (\vec{v} \cdot \nabla)\frac{\partial v}{\partial y} + (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) v + f\frac{\partial u}{\partial y} + \frac{\mathrm{d}f}{\mathrm{d}y}u = -\alpha \frac{\partial^2 p}{\partial y^2} - \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial y} \quad (6)$$

## For the vorticity equation, take (4)-(5)

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + (\vec{v} \cdot \nabla)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + (\frac{\partial \vec{v}}{\partial x} \cdot \nabla) v - (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\mathrm{d}f}{\mathrm{d}y}v$$

$$= \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x} - \frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} \quad (7)$$

Let $\quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \ \delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

Note $\quad \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x} - \frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} = (\nabla p \times \nabla \alpha) \cdot \vec{k}$

Equation (7) becomes,

$$\frac{\partial \zeta}{\partial t} + (\vec{v} \cdot \nabla)\zeta + (\frac{\partial \vec{v}}{\partial x} \cdot \nabla) v - (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) u + f\delta + \frac{\mathrm{d}f}{\mathrm{d}y}v = (\nabla p \times \nabla \alpha) \cdot \vec{k} \quad (8)$$

$$\frac{\partial(\zeta + f)}{\partial t} + (\vec{v} \cdot \nabla)(\zeta + f) + (\frac{\partial \vec{v}}{\partial x} \cdot \nabla) v - (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) u + f\delta = (\nabla p \times \nabla \alpha) \cdot \vec{k} \tag{9}$$

$$\frac{\mathrm{d}(\zeta + f)}{\mathrm{d}t} + (\frac{\partial \vec{v}}{\partial x} \cdot \nabla) v - (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) u + f\delta = (\nabla p \times \nabla \alpha) \cdot \vec{k} \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

<u>For the divergence equation, take (3)+(6)</u>

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + (\vec{v} \cdot \nabla)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + (\frac{\partial \vec{v}}{\partial x} \cdot \nabla) u + (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) v - f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\mathrm{d}f}{\mathrm{d}y} u$$

$$= -\alpha \frac{\partial^2 p}{\partial y^2} - \alpha \frac{\partial^2 p}{\partial x^2} - \frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial y} \tag{11}$$

Note $\quad \frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial y} = (\nabla_h \alpha) \cdot (\nabla_h p)$

where $\quad \nabla_h = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix}$

$$\frac{\partial \delta}{\partial t} + (\vec{v} \cdot \nabla)\delta + (\frac{\partial \vec{v}}{\partial x} \cdot \nabla) u + (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) v - f\zeta + \frac{\mathrm{d}f}{\mathrm{d}y} u = -\alpha \nabla_h^2 p - (\nabla_h \alpha) \cdot (\nabla_h p) \tag{12}$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} + (\frac{\partial \vec{v}}{\partial x} \cdot \nabla) u + (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) v - f\zeta + \frac{\mathrm{d}f}{\mathrm{d}y} u = -\alpha \nabla_h^2 p - (\nabla_h \alpha) \cdot (\nabla_h p) \tag{13}$$

<u>Note the following terms that can be rewritten in (10) and (13)</u>

$$(\frac{\partial \vec{v}}{\partial x} \cdot \nabla) v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} \tag{14}$$

$$(\frac{\partial \vec{v}}{\partial y} \cdot \nabla) u = \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} \tag{15}$$

$$(\frac{\partial \vec{v}}{\partial x} \cdot \nabla) u = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} \tag{16}$$

$$(\frac{\partial \vec{v}}{\partial y} \cdot \nabla) v = \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} \tag{17}$$

For use in (10), take (14)-(15)

$$\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}$$

$$= \zeta\delta - \left(\xi \frac{\partial w}{\partial x} + \eta \frac{\partial w}{\partial y}\right) \tag{18}$$

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

For use in (13), take (16)+(17). This can be written as

$$\begin{aligned}
(\frac{\partial \vec{v}}{\partial x} \cdot \nabla) u + (\frac{\partial \vec{v}}{\partial y} \cdot \nabla) v &= \left((\nabla_h \vec{v}) \nabla\right) \cdot \vec{v}_h \\
&= \left( \left( \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} (u \ v \ w) \right) \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \right) \cdot \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \left( \begin{pmatrix} \partial u/\partial x & \partial v/\partial x & \partial w/\partial x \\ \partial u/\partial y & \partial v/\partial y & \partial w/\partial y \end{pmatrix} \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \right) \cdot \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial u}{\partial x}\frac{\partial}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial}{\partial z} \\ \frac{\partial u}{\partial y}\frac{\partial}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z}
\end{aligned}$$

## The vorticity and divergence equations

$$\frac{\mathrm{d}(\zeta + f)}{\mathrm{d}t} + (\zeta + f)\delta - \left(\xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y}\right) = (\nabla p \times \nabla\alpha) \cdot \vec{k} \tag{19}$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} + \left((\nabla_h \vec{v}) \nabla\right) \cdot \vec{v}_h - f\zeta + \frac{\mathrm{d}f}{\mathrm{d}y}u = -\alpha\nabla_h^2 p - (\nabla_h \alpha) \cdot (\nabla_h p) \tag{20}$$