THE DOUBLE PENDULUM

NUMERICAL SOLUTION IN THE LAGRANGE FORMULATION

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The double pendulum consists of two sections. Suspended from a fixed point is a mass, m_1 constrained to be a distance l_1 from that point, but is allowed to vary its orientation (θ_1 from the vertical). A second mass, m_2 is suspended a further distance l_2 from the first mass and it too has an angular degree of freedom, being an angle θ_2 from the vertical at any instant. The physical system is illustrated in Fig. 1 (the system is in a gravitational field of magnitude g, directed downwards).

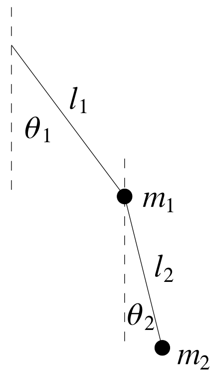

Figure 1: The double pendulum set-up. The lengths l_1 and l_2, and the masses m_1 and m_2 are fixed. The system has the degrees of freedom θ_1 and θ_2.

1. CONSTRUCT THE LAGRANGIAN

The Lagrangian is defined as $L = T - V$, where T and V are expressions for the kinetic and potential energies respectively. In the Lagrange formulation, L satisfies Lagrange's equations which determine the time evolution of the system. These are given later. The components of L are,

$$V = -gm_1 l_1 \cos\theta_1 - gm_2 \left(l_1 \cos\theta_1 + l_2 \cos\theta_2\right), \tag{1.1}$$

$$T = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2). \tag{1.2}$$

In Eq. (1.2), the x and y refer to the positions of the respective masses (the dots refer to their rates of change and so the choice of origin is irrelevant). We require Eq. (1.2) to be written in terms of angles and so we exploit the transformations (we now choose the origin to be the fixed point),

$$x_1 = l_1 \sin\theta_1, \qquad \dot{x}_1 = l_1 \cos\theta_1 \dot{\theta}_1, \tag{1.3}$$

$$y_1 = -l_1 \cos\theta_1, \qquad \dot{y}_1 = l_1 \sin\theta_1 \dot{\theta}_1, \tag{1.4}$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2, \qquad \dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2, \tag{1.5}$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2, \qquad \dot{y}_2 = l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2. \tag{1.6}$$

Inserting Eqs. (1.3) to (1.6) into Eq. (1.2) (and simplifying), the Lagrangian becomes,

$$L = \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$
$$+ g m_1 l_1 \cos\theta_1 + g m_2 \left(l_1 \cos\theta_1 + l_2 \cos\theta_2 \right). \qquad 1.7$$

2. THE EQUATIONS OF MOTION

For each degree of freedom, there is an equation of motion. For degree i this is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) = \frac{\partial L}{\partial \theta_i}, \qquad 2.1$$

where in the case of the double pendulum, $i = 1,2$. The θ_i and the $\dot{\theta}_i$ represent separate variables. Eqs. (2.1) are coupled second order equations. This is apparent if the total time derivative is expanded by the chain rule,

$$\sum_j \frac{\partial^2 L}{\partial \dot{\theta}_i \partial \theta_j}\dot{\theta}_j + \sum_j \frac{\partial^2 L}{\partial \dot{\theta}_i \partial \dot{\theta}_j}\ddot{\theta}_j = \frac{\partial L}{\partial \theta_i}, \qquad 2.2$$

where $j = 1,2$. The sets of partial derivatives can be represented as matrices **A** and **B**. Let,

$$\mathbf{A}_{ij} = \frac{\partial^2 L}{\partial \dot{\theta}_i \partial \theta_j} \quad \text{and} \quad \mathbf{B}_{ij} = \frac{\partial^2 L}{\partial \dot{\theta}_i \partial \dot{\theta}_j}, \qquad 2.3$$

(the components of these matrices, and the right-hand-sides of Eqs. (2.2) are given in the appendix). Equations 2.2 can then be written as,

$$\mathbf{A}\begin{pmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{pmatrix} + \mathbf{B}\begin{pmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{pmatrix} = \begin{pmatrix}\partial L/\partial \theta_1 \\ \partial L/\partial \theta_2\end{pmatrix}, \qquad 2.4$$

with the different i represented by the different rows of this matrix equation.

3. TRANSFORMING TO FIRST ORDER

For the purposes of numerical solution, the equations of motion are best represented as a set of first order equations. Let $\alpha_i = \dot{\theta}_i$. This has the consequence that $\dot{\alpha}_i = \ddot{\theta}_i$. Writing the state vector with the four components θ_1, θ_2, α_1 and α_2, and the equation set becomes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\theta_1 \\ \theta_2 \\ \alpha_1 \\ \alpha_2\end{pmatrix} = \left(\begin{array}{c|c} 0 & \mathbf{I} \\ \hline 0 & -\mathbf{B}^{-1}\mathbf{A} \end{array}\right)\begin{pmatrix}\theta_1 \\ \theta_2 \\ \hline \alpha_1 \\ \alpha_2\end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathbf{B}^{-1}\begin{pmatrix}\partial L/\partial \theta_1 \\ \partial L/\partial \theta_2\end{pmatrix}\end{pmatrix}. \qquad 3.1$$

In Eq. (3.1), each quadrant of the large matrix is itself a 2×2 matrix. The top two rows are simply a definition of the change of variables, and the bottom two rows are the Lagrangian equations of motion (Eq. (2.4) rearranged). The inverse of **B** is easy to calculate as the matrix is of only small rank (see the appendix).

4. RUNGE-KUTTA SOLUTION

For convenience we will write Eq. (3.1) in the simplified form of,

$$\dot{\mathbf{y}} = \mathbf{T}_\mathbf{y}\mathbf{y} + \mathbf{g}_\mathbf{y}, \tag{4.1}$$

where the vector **y** represents the four variables pertaining to the angles and their rates of change. The 4×4 matrix **T** and the vector **g** are functions of the state variable **y** (denoted by the subscript). We will solve the equations using the standard fourth order Runge-Kutta solution [1]. This has five stages,

1. Calculate $\mathbf{k}_1$: $\mathbf{k}_1 = \Delta t\left(\mathbf{T}_\mathbf{y}\mathbf{y} + \mathbf{g}_\mathbf{y}\right)$
2. Calculate $\mathbf{k}_2$: $\mathbf{k}_2 = \Delta t\left(\mathbf{T}_{\mathbf{y}+\mathbf{k}_1/2}(\mathbf{y} + \mathbf{k}_1/2) + \mathbf{g}_{\mathbf{y}+\mathbf{k}_1/2}\right)$
3. Calculate $\mathbf{k}_3$: $\mathbf{k}_3 = \Delta t\left(\mathbf{T}_{\mathbf{y}+\mathbf{k}_2/2}(\mathbf{y} + \mathbf{k}_2/2) + \mathbf{g}_{\mathbf{y}+\mathbf{k}_2/2}\right)$
4. Calculate $\mathbf{k}_4$: $\mathbf{k}_4 = \Delta t\left(\mathbf{T}_{\mathbf{y}+\mathbf{k}_3}(\mathbf{y} + \mathbf{k}_3) + \mathbf{g}_{\mathbf{y}+\mathbf{k}_3}\right)$
5. Calculate $\mathbf{y}(t + \Delta t)$: $\mathbf{y}(t + \Delta t) = \mathbf{y}(t) + \mathbf{k}_1/6 + \mathbf{k}_2/3 + \mathbf{k}_3/3 + \mathbf{k}_4/6$

The value $\mathbf{y}(t = 0)$ is set by the initial conditions (as specified).

5. APPENDIX

We note the partial derivatives of L (Eq. (1.7)),

$$\frac{\partial^2 L}{\partial\dot{\theta}_1\partial\theta_1} = -m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2), \tag{5.1}$$

$$\frac{\partial^2 L}{\partial\dot{\theta}_1\partial\theta_2} = m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2), \tag{5.2}$$

$$\frac{\partial^2 L}{\partial\dot{\theta}_2\partial\theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2), \tag{5.3}$$

$$\frac{\partial^2 L}{\partial\dot{\theta}_2\partial\theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2), \tag{5.4}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_1^2} = (m_1 + m_2)\, l_1^2, \tag{5.5}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_1 \partial \dot{\theta}_2} = m_2 l_1 l_2 \cos(\theta_1 - \theta_2), \tag{5.6}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_2 \partial \dot{\theta}_1} = m_2 l_1 l_2 \cos(\theta_1 - \theta_2), \tag{5.7}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_2^2} = m_2 l_2^2, \tag{5.8}$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g(m_1 + m_2)\, l_1 \sin\theta_1, \tag{5.9}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g m_2 l_2 \sin\theta_2. \tag{5.10}$$

Noteworthy also is the inverse of matrix **B**. If,

$$\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{5.11}$$

then the inverse is,

$$\mathbf{B}^{-1} = \frac{1}{B_{11}B_{22} - B_{12}B_{21}} \begin{pmatrix} B_{22} & -B_{12} \\ -B_{21} & B_{11} \end{pmatrix}. \tag{5.12}$$

6. REFERENCE

[1] Press W.H., Teukolsky S.A., Vettering W.T., Flannery B.P., Numerical Recipes in Fortran (section 16.1), Cambridge University Press (1994).