# PDF symmetry of difference between two iid variables

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Suppose that two variables $x_A$ and $x_B$ are drawn from independent and identical (but otherwise arbitrary) distributions. Let this common distribution be $p(x)$:

$$\begin{array}{lll} x_A & \text{drawn from} & p(x_A), \\ x_B & \text{drawn from} & p(x_B). \end{array}$$

Ingelby et al (2012) (Sec. 2.3) states that if two variables are drawn from identical distributions then their differences must be from a symmetric distribution. No condition of symmetry is imposed on the identical distributions, $p(x)$. A number of questions arises:

1. How do we prove that this is correct?
2. What is this symmetric distribution in terms of $p(x)$?

Let the difference be $y = x_B - x_A$ and let the probability of getting a particular difference, $y$, for a specific set of variables $x_A$ and $x_B$ be $p_{\text{diff}}(y, x_A, x_B)$:

$$p_{\text{diff}}(y, x_A, x_B) = p(x_A)p(x_B)\delta(x_A - x_B - y).$$

We want to know the probability of this difference, $y$, without the conditioning on $x_A$ and $x_B$. To calculate this (call this $p_{\text{diff}}(y)$), integrate over all values of $x_A$ and $x_B$:

$$\begin{aligned} p_{\text{diff}}(y) &= \int\int dx_A dx_B p_{\text{diff}}(y, x_A, x_B), \\ &= \int\int dx_A dx_B p(x_A)p(x_B)\delta(x_A - x_B - y), \\ &= \int\int dx\, p(x)p(x - y), \end{aligned}$$

where we have relabelled $x_B$ as $x$. Now let $v = x - y/2$ be an alternative variable for $x$. Then

$$\begin{aligned} dv &= dx, \\ x &= v + y/2, \\ x - y &= v - y/2, \end{aligned}$$

$p_{\text{diff}}(y)$ is then equivalent to

$$p_{\text{diff}}(y) = \int \int dv\, p(v + y/2)p(v - y/2).$$

The important property of this last result is that $p_{\text{diff}}(y) = p_{\text{diff}}(-y)$, which is a symmetric distribution. It also says what the form of the symmetric distribution is.

Ingleby N.B., Lorenc A.C., Ngan K., Rawlins F., Jackson D.R., Improved variational analyses using a non-linear humidity control variable, QJRMS DOI:10.1002/qj.2073 (2012).