NOTES AND QUESTIONS ON VARIATIONAL CALCULUS

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1. MINIMIZING FUNCTIONS SUBJECT TO CONSTRAINTS

1.1 Two degrees of freedom, one constraint

Problem: find the stationary point of $f(x,y)$ subject to the constraint $g(x,y) = 0$.

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \qquad (1.1,\ 1.2)$$

The single Lagrange multiplier, λ, is associated with the constraint.

1.2 N degrees of freedom, M constraints

Problem: find the stationary point of $f(x_1, x_2, \ldots, x_N)$ subject to the constraints $g_m(x_1, x_2, \ldots, x_N) = 0$ for $1 \leqslant m \leqslant M$.

$$\frac{\partial}{\partial x_n}\left(f + \sum_{m=1}^{M} g_m \lambda_m\right) = 0, \qquad 1 \leqslant n \leqslant N. \qquad (1.3)$$

This set of equations can be written as a vector equation with each vector component representing one particular 'n' from the above,

$$\nabla f + \mathbf{G}^T \vec{\lambda} = 0, \qquad (1.4)$$

where the $M \times N$ matrix $\mathbf{G}$ has elements,

$$G_{mn} = \frac{\partial G_m}{\partial x_n}. \qquad (1.5)$$

There are as many Lagrangian multipliers as there are constraints. "The Lagrange multiplier can be interpreted as a measure of the sensitivity of the value of the function f at the stationary point to changes in the constraint." (Daley p. 245). Differentiate Eq. (1.3) with respect to the constraint g_k,

$$\frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial g_k} + \sum_{m=1}^{M} \frac{\partial g_m}{\partial g_k} \lambda_m\right) = 0. \qquad (1.6)$$

Noting that $\partial g_m / \partial g_k = \delta_{mk}$,

$$\frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial g_k} + \lambda_k\right) = 0. \qquad (1.7)$$

We would like to set the term in brackets directly equal to zero to prove the statement in Daley, but what about the differential operator?

2. MINIMIZING FUNCTIONALS

2.1 One variable, one function, up to first derivative of function involved in functional

This section is involved in finding a function that makes a functional stationary. In this example, we ask: what $u(x)$, makes the following functional stationary,

$$I\{u(x)\} = \int_{x_a}^{x_b} F(x, u, u')\, dx, \tag{2.1}$$

where $u' \equiv \mathrm{d}u/\mathrm{d}x$. The equation which must be satisfied at the stationary point is the *Euler-Lagrange equation*,

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} = 0. \tag{2.2}$$

This Euler-Lagrange equation is derived in appendix A.1.

2.2 One variable, two functions, up to first derivative of functions involved in functional

An extension from the first case is to include a functional dependence from two functions,

$$I\{u(x), v(x)\} = \int_{x_a}^{x_b} F(x, u, u', v, v')\, dx. \tag{2.3}$$

At the point that this functional is stationary, two Euler-Lagrange equations are satisfied,

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} = 0 \quad \text{and} \quad \frac{\partial F}{\partial v} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial v'} = 0. \tag{2.4}$$

2.3 Two variables, one function, up to first derivative of function involved in functional

Allowing the single function now to depend on two variables, $u(x,y)$, gives a functional of the following form,

$$I\{u(x, y)\} = \int F(x, y, u, u_x', u_y')\, dx\, dy, \tag{2.5}$$

where $u_x' \equiv \mathrm{d}u/\mathrm{d}x$ and $u_y' \equiv \mathrm{d}u/\mathrm{d}y$, and the integral is over some pre-defined area in x, y space. The Euler-Lagrange equation for this functional has an extra term from Eq. (2.2),

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u_x'} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial F}{\partial u_y'} = 0. \tag{2.6}$$

2.4 One variable, one function, up to first derivative of function involved in functional, one constraint

This section builds on Eq. (2.1) which is for the same case as this but with no constraint imposed on the extremum. Let the constraint be expressed as the integral,

$$\int_{x_a}^{x_b} g(x,\, u,\, u')\, dx \;=\; 0. \tag{2.7}$$

The functional that is then minimized is then (c.f. Eq. (2.1)),

$$I\{u(x)\} \;=\; \int_{x_a}^{x_b} \left[F(x,\, u,\, u') \,+\, \lambda g(x,\, u,\, u')\right] dx. \tag{2.8}$$

The Euler-Lagrange equation that is satisfied at an extremum of this functional is (c.f. Eq. (2.2)),

$$\frac{\partial (F + \lambda g)}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial (F + \lambda g)}{\partial u'} = 0.$$

APPENDIX A

Appendix A.1 - Derivation of the Euler-Lagrange equation for a functional of one function up to the first derivative (as in section 2.1).

The functional that we wish to make stationary is given as Eq. (2.1). Suppose that we know that the functional is stationary at $u(x)$. Consider the slightly perturbed function, $\tilde{u}(x)$,

$$\tilde{u}(x) \;=\; u(x) \,+\, \varepsilon\eta(x), \tag{A.1}$$

where the perturbing function, $\eta(x) \to 0$ at each end of the integration domain ($x = x_a$ and x_b, as defined in Eq. (2.1)). From Eq. (2.1), the functional, I, is made stationary when $\delta I = 0$. δI is made up of contributions from differences in F over all locations inside the domain,

$$\delta I \;=\; \int_{x_a}^{x_b} \delta F\, dx, \tag{A.2}$$

and δF can be expressed as having contributions from changes in $u(x)$ and $u'(x)$,

$$\delta F \;=\; \frac{\partial F}{\partial u}\delta u \,+\, \frac{\partial F}{\partial u'}\delta u'. \tag{A.3}$$

In the vicinity of the stationary function, $u(x)$, Eq. (A.1) allows this chain rule to be written as,

$$\delta F \;=\; \varepsilon\left(\frac{\partial F}{\partial u}\eta \,+\, \frac{\partial F}{\partial u'}\eta'\right). \tag{A.4}$$

Inserting this into Eq. (A.2),

$$\delta I \;=\; \varepsilon \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial u}\eta \,+\, \frac{\partial F}{\partial u'}\eta'\right) dx \;=\; 0. \tag{A.5}$$

The second term in the integral must be tackled via an integration by parts<,

$$\int_{x_a}^{x_b} \frac{\partial F}{\partial u'} \frac{\mathrm{d}\eta}{\mathrm{d}x} dx = \left[\frac{\partial F}{\partial u'} \eta \right]_{x_a}^{x_b} - \int_{x_a}^{x_b} \eta \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial u'} dx. \tag{A.6}$$

Noting that, due to the nature of $\eta(x)$ at the boundaries, the first term on the right hand side of Eq. (A.6) is zero, the integral for δI equal to,

$$\delta I = \varepsilon \int_{x_a}^{x_b} \eta \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial u'} \right) dx = 0. \tag{A.7}$$

Equation (A.7) should be true for an arbitrary function $\eta(x)$, and so it must be the case that the term within brackets is itself zero. This is the Euler-Lagrange equation as in Eq. (2.2).