# If a covariance model gives separable structure functions in real space, does it also give separable structure functions in spectral space?

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## Real space

A homogeneous error covariance model that gives separable structure functions in real space can be written as a convolution. A convolution in 1d has the form,

$$g(x) = \int \mathrm{d}x' C(x - x') f(x'), \tag{1}$$

where $C(x)$ is the function that we are convoluting with, (the correlation function) and $f(x)$ is the function that we are convoluting. If, for instance, $f(x) = \delta(x - x_0)$, then the result of Eq. (1),

$$g(x) = \int \mathrm{d}x' C(x - x') \delta(x' - x_0),$$

$$= C(x - x_0), \tag{2}$$

is just a shift of $C(x)$ to the position of the delta-function.

Let our 2d error covariance model involve the convolution with the 2d separable correlation function,

$$H(\phi) V(z), \tag{3}$$

where $\phi$ is the horizontal position and $z$ is the vertical position. The real-space correlation model is thus,

$$g(\phi, z) = \int \int \mathrm{d}\phi' \, \mathrm{d}z' H(\phi - \phi') V(z - z') f(\phi', z'). \tag{4}$$

The convolution is just a linear operator, which can be written in the more familiar way using the matrix notation,

$$\vec{g} = \mathbf{C}\vec{f}. \tag{5}$$

## Spectral space

In spectral space, we first write the fields as linear combinations of horizontal and vertical basis functions. For $g$,

$$g(\phi, z) = \int \int \mathrm{d}k' \, \mathrm{d}\nu' \bar{g}(k', \nu') F_h(\phi, k') F_v(z, \nu'), \tag{6}$$

and similarly for $f$,

$$f(\phi, z) = \int \int \mathrm{d}k' \, \mathrm{d}\nu' \bar{f}(k', \nu') F_h(\phi, k') F_v(z, \nu'). \tag{7}$$

$k$ is the horizontal mode number (e.g. for Fourier mode), and $\nu$ is the vertical mode number. Note the orthogonality relations of the horizontal and vertical basis functions,

$$\int \mathrm{d}\phi F_h^*(\phi, k) F_h(\phi, k') = \delta(k - k'), \tag{8}$$

$$\int \mathrm{d}z F_v^*(z, \nu) F_v(z, \nu') = \delta(\nu - \nu'). \tag{9}$$

Inserting Eqs. (6) and (7) into Eq. (4) gives,

$$\int \int \mathrm{d}k' \, \mathrm{d}\nu' \bar{g}(k', \nu') F_h(\phi, k') F_v(z, \nu') =$$

$$\int\int \mathrm{d}\phi'\,\mathrm{d}z' H(\phi-\phi')V(z-z')\int\int \mathrm{d}k'\,\mathrm{d}\nu'\bar{f}(k',\nu')F_h(\phi,k')F_v(z,\nu'). \quad (10)$$

Now using the orthogonality relations, Eqs. (8) and (9) by multiplying Eq. (10) by the starred variables in Eqs. (8) and (9) and integrating over horizontal and vertical positions,

$$\int\int \mathrm{d}\phi\,\mathrm{d}z F_h^*(\phi,k)F_v^*(z,\nu)\int\int \mathrm{d}k'\,\mathrm{d}\nu'\bar{g}(k',\nu')F_h(\phi,k')F_v(z,\nu') =$$

$$\int\int \mathrm{d}\phi\,\mathrm{d}z F_h^*(\phi,k)F_v^*(z,\nu)\int\int \mathrm{d}\phi'\,\mathrm{d}z' H(\phi-\phi')V(z-z')\int\int \mathrm{d}k'\,\mathrm{d}\nu'\bar{f}(k',\nu')F_h(\phi,k')F_v(z,\nu'). \quad (11)$$

Orthogonality results in the left hand side simplifying,

$$\bar{g}(k,\nu) = \int\int \mathrm{d}\phi\,\mathrm{d}z F_h^*(\phi,k)F_v^*(z,\nu)\int\int \mathrm{d}\phi'\,\mathrm{d}z' H(\phi-\phi')V(z-z')\int\int \mathrm{d}k'\,\mathrm{d}\nu'\bar{f}(k',\nu')F_h(\phi,k')F_v(z,\nu'). \quad (12)$$

This is the correlation model of Eq. (4), but in spectral space. Let us rearrange the order of the integrals in Eq. (12),

$$\bar{g}(k,\nu) = \int\int \mathrm{d}k'\,\mathrm{d}\nu'\Big(\int\int \mathrm{d}\phi\,\mathrm{d}z F_h^*(\phi,k)F_v^*(z,\nu)\int\int \mathrm{d}\phi'\,\mathrm{d}z' H(\phi-\phi')V(z-z')F_h(\phi,k')F_v(z,\nu')\Big)\bar{f}(k',\nu'). \quad (13)$$

The terms in brackets constitute the covariance model operator in spectral space. The question now is: is this model separable in spectral space? This is very easy to show by acting with Eq. (13) on the state,

$$\bar{f}(k',\nu') = \delta(k'-k_0)\delta(\nu'-\nu_0). \quad (14)$$

If the result, $\bar{g}(k,\nu)$ is separable, then we have proved the result,

$$\bar{g}(k,\nu) = \int\int \mathrm{d}\phi\,\mathrm{d}z F_h^*(\phi,k)F_v^*(z,\nu)\int\int \mathrm{d}\phi'\,\mathrm{d}z' H(\phi-\phi')V(z-z')F_h(\phi,k_0)F_v(z,\nu_0),$$

$$= \int \mathrm{d}\phi F_h^*(\phi,k)\int \mathrm{d}\phi' H(\phi-\phi')F_h(\phi,k_0)\times\int \mathrm{d}z F_v^*(z,\nu)\int \mathrm{d}z' V(z-z')F_v(z,\nu_0). \quad (15)$$

This is indeed separable - Eq. (15) is the product of a function that depends upon $k$, but not $\nu$, with another that depends upon $\nu$, but not $k$.

## Conclusion

A correlation mode that is homogeneous and separable in real space is also separable in spectral space. Note that eq. (15) is not necessarily homogeneous in spectral space, ie the first product depends upon $k_0$ and the second product depends upon $\nu_0$ (but note that the nature of the inhomogenaities may not be arbitrary - this still needs to be proved). By symmetry, doing the analysis in reverse (spectral to real spaces), a correlation model that is homogeneous and separable in spectral space is also separable in real space, but not necessarily homogeneous.