Particle Filter Formulae

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Bayes theorem ('prior' to 'posterior')

$$p(\psi^n \mid d^n) = \frac{1}{A^n} p(d^n \mid \psi^n) p(\psi^n), \tag{1}$$

time level n.

'Prior' in particle form

$$p(\psi^n) = \frac{1}{N} \sum_{i=1}^{N} w_i^n \delta(\psi^n - \psi_i^n), \tag{2}$$

$$\sum_{i=1}^{N} w_i^n = N.$$

'Posterior' in particle form

$$p(\psi^n \mid d^n) = \frac{1}{N} \sum_{i=1}^{N} \tilde{w}_i^n \delta(\psi^n - \psi_i^n), \tag{3}$$

$$\sum_{i=1}^{N} \tilde{w}_i^n = N.$$

Bayes theorem in particle form

Substitute (2) and (3) in (1)

$$\frac{1}{N} \sum_{i=1}^{N} \tilde{w}_i^n \delta(\psi^n - \psi_i^n) = \frac{1}{A^n} p(d^n \mid \psi^n) \frac{1}{N} \sum_{i=1}^{N} w_i^n \delta(\psi^n - \psi_i^n),$$

Multiply by $\delta(\psi^n - \psi_{i'}^n)$ and integrate over all ψ^n

$$\frac{1}{N} \sum_{i=1}^{N} \tilde{w}_i^n \int d\psi^n \, \delta(\psi^n - \psi_i^n) \delta(\psi^n - \psi_{i'}^n) =$$

$$\frac{1}{A^n} \int d\psi^n \, p(d^n \mid \psi^n) \frac{1}{N} \sum_{i=1}^{N} w_i^n \delta(\psi^n - \psi_i^n) \delta(\psi^n - \psi_{i'}^n),$$

$$\therefore \tilde{w}_i^n = \frac{1}{A^n} p(d^n \mid \psi_i^n) w_i^n. \tag{4}$$

Particles don't change their positions in this step - just their weights.

Transition density ('forecast' of 'prior')

$$p(\psi^n) = \int d\psi^{n-1} \, p(\psi^n \mid \psi^{n-1}) p(\psi^{n-1}), \tag{5}$$

where $p(\psi^n \mid \psi^{n-1})$ is the transition density. For a deterministic model

$$\psi^n = f(\psi^{n-1}),$$

the transition density is a delta-function

$$p(\psi^n \mid \psi^{n-1}) = \delta(\psi^n - f(\psi^{n-1})),$$

$$\therefore p(\psi^n) = \int d\psi^{n-1} \delta(\psi^n - f(\psi^{n-1})) p(\psi^{n-1}). \tag{5a}$$

For a stochastic model

$$\psi^n = f(\psi^{n-1}) + \beta^n,$$

where β^n is a random variable with distribution $r(\beta^n)$, the transition density is

$$p(\psi^n \mid \psi^{n-1}) = r(\psi^n - f(\psi^{n-1})),$$

$$\therefore p(\psi^n) = \int d\psi^{n-1} r(\psi^n - f(\psi^{n-1})) p(\psi^{n-1}). \qquad (5b)$$

Proposal density

The transition density may be written as follows

$$p(\psi^n \mid \psi^{n-1}) = q(\psi^n \mid \psi^{n-1}, d^n) \frac{p(\psi^n \mid \psi^{n-1})}{q(\psi^n \mid \psi^{n-1}, d^n)}, \qquad (6)$$

where $q(\psi^n \mid \psi^{n-1}, d^n)$ is a chosen function (called the proposal density) that has support greater than or equal to that of $p(\psi^n \mid \psi^{n-1})$. It is possible to replace the transition density with the proposal density

$$p(\psi^n \mid \psi^{n-1}) \rightarrow q(\psi^n \mid \psi^{n-1}, d^n), \qquad (7)$$

as long as the compensation term, $p(\psi^n \mid \psi^{n-1}) / q(\psi^n \mid \psi^{n-1}, d^n)$, multiplies $p(\psi^{n-1})$

$$p(\psi^{n-1}) \rightarrow \frac{p(\psi^n \mid \psi^{n-1})}{q(\psi^n \mid \psi^{n-1}, d^n)}) p(\psi^{n-1}). \qquad (8)$$

This can be seen by substituting (6) into (5) to give

$$p(\psi^n) = \int d\psi^{n-1} q(\psi^n \mid \psi^{n-1}, d^n) \left(\frac{p(\psi^n \mid \psi^{n-1})}{q(\psi^n \mid \psi^{n-1}, d^n)}) p(\psi^{n-1}) \right). \qquad (9)$$

$q(\psi^n \mid \psi^{n-1}, d^n)$ may be chosen to depend upon the observations if required.

The equivalent picture in terms of particle weights emerges by substituting (2) into (8)

$$\sum_{i=1}^{N} w_i^{n-1} \delta(\psi^{n-1} - \psi_i^{n-1}) \rightarrow \frac{p(\psi^n \mid \psi^{n-1})}{q(\psi^n \mid \psi^{n-1}, d^n)} \sum_{i=1}^{N} w_i^{n-1} \delta(\psi^{n-1} - \psi_i^{n-1}),$$

multiplying by $\delta(\psi^{n-1} - \psi_{i'}^{n-1})$ and integrating over all ψ^{n-1}

$$\sum_{i=1}^{N} w_i^{n-1} \int d\psi^{n-1} \, \delta(\psi^{n-1} - \psi_i^{n-1}) \, \delta(\psi^{n-1} - \psi_{i'}^{n-1}) \rightarrow$$

$$\int d\psi^{n-1} \frac{p(\psi^n \mid \psi^{n-1})}{q(\psi^n \mid \psi^{n-1}, d^n)} \sum_{i=1}^{N} w_i^{n-1} \delta(\psi^{n-1} - \psi_i^{n-1}) \, \delta(\psi^{n-1} - \psi_{i'}^{n-1}),$$

$$\therefore w_i^{n-1} \rightarrow \frac{p(\psi^n \mid \psi_i^{n-1})}{q(\psi^n \mid \psi_i^{n-1}, d^n)} w_i^{n-1}. \qquad (10)$$