Proof of identity (involving forecast and analysis error covariances)

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March 29, 2019

Aim

Prove that the following identity holds:

$$\left(\mathbf{R} + \mathbf{H}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\right)^{-1} = \mathbf{R}^{-1}\left(\mathbf{R} - \mathbf{H}\mathbf{P}^{\mathrm{a}}\mathbf{H}^{\mathrm{T}}\right)\mathbf{R}^{-1}, \tag{1}$$

where $\mathbf{R}$ is the observation error covariance matrix, $\mathbf{H}$ is the observation operator, $\mathbf{P}^{\mathrm{f}}$ is the forecast error covariance matrix, and $\mathbf{P}^{\mathrm{a}}$ is the analysis error covariance matrix.

Proof

The following is the usual form of the Sherman-Morrison-Woodbury formula:

$$\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\left(\mathbf{R} + \mathbf{H}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{P}^{\mathrm{f}\,-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}. \tag{2}$$

Since $\mathbf{P}^{\mathrm{f}}$ in this identity is really an arbitrary invertible square matrix, we may replace $\mathbf{P}^{\mathrm{f}} \to -\mathbf{P}^{\mathrm{a}}$ in (2):

$$-\mathbf{P}^{\mathrm{a}}\mathbf{H}^{\mathrm{T}}\left(\mathbf{R} - \mathbf{H}\mathbf{P}^{\mathrm{a}}\mathbf{H}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H} - \mathbf{P}^{\mathrm{a}\,-1}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}, \tag{3}$$

or rearranged:

$$-\left(\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H} - \mathbf{P}^{\mathrm{a}\,-1}\right)\mathbf{P}^{\mathrm{a}}\mathbf{H}^{\mathrm{T}} = \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathbf{R} - \mathbf{H}\mathbf{P}^{\mathrm{a}}\mathbf{H}^{\mathrm{T}}\right). \tag{4}$$

We also know the relationship between $\mathbf{P}^{\mathrm{a}}$ and $\mathbf{P}^{\mathrm{f}}$:

$$\mathbf{P}^{\mathrm{a}\,-1} = \mathbf{P}^{\mathrm{f}\,-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}, \tag{5}$$

which is the Hessian of the problem. We will now work with (1) (after acting on each side with $\mathbf{H}^{\mathrm{T}}$) and show that it is correct.

$$\begin{aligned}
\mathbf{H}^{\mathrm{T}}\left(\mathbf{R} + \mathbf{H}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\right)^{-1} &= \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathbf{R} - \mathbf{H}\mathbf{P}^{\mathrm{a}}\mathbf{H}^{\mathrm{T}}\right)\mathbf{R}^{-1} \\
\text{use (4):} \quad &= -\left(\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H} - \mathbf{P}^{\mathrm{a}\,-1}\right)\mathbf{P}^{\mathrm{a}}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1} \\
\text{use (5):} \quad &= -\left(\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H} - \mathbf{P}^{\mathrm{f}\,-1} - \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right)\left(\mathbf{P}^{\mathrm{f}\,-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1} \\
&= \mathbf{P}^{\mathrm{f}\,-1}\left(\mathbf{P}^{\mathrm{f}\,-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1} \\
\text{use (2):} \quad &= \mathbf{P}^{\mathrm{f}\,-1}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\left(\mathbf{R} + \mathbf{H}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\right)^{-1} \\
&= \mathbf{H}^{\mathrm{T}}\left(\mathbf{R} + \mathbf{H}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\right)^{-1}.
\end{aligned} \tag{6}$$

Q.E.D.