Summary of the 'T' transform in the PV scheme

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1. Definition of transforms

All variables are increments and all operators are linear:

$$\vec{x} = \mathbf{U}\vec{v}, \quad (1)$$

$$\vec{y} = \mathbf{A}\vec{x}, \quad (2)$$

$$\vec{v} = \mathbf{T}\vec{x}, \quad (3)$$

$$\therefore \mathbf{A}\vec{x} = \mathbf{A}\mathbf{U}\vec{v}, \quad (4)$$

$$\therefore \vec{v} = (\mathbf{A}\mathbf{U})^{-1}\mathbf{A}\vec{x}, \quad (5)$$

$$\therefore \mathbf{T} = (\mathbf{A}\mathbf{U})^{-1}\mathbf{A}, \quad (6)$$

$$\vec{v} = \begin{pmatrix} s \\ {}^{u}p \\ \chi \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \qquad \vec{y} = \begin{pmatrix} PV \\ \bar{PV} \\ \nabla \cdot \vec{u} \end{pmatrix}. \quad (7abc)$$

$\vec{x}$ is a vector of model variables, $\vec{v}$ is a vector of control parameters (s is balanced streamfunction, ${}^{u}p$ is unbalanced pressure and χ is the velocity potential), and $\vec{y}$ is a vector of associated parameters (PV is potential vorticity, $\bar{PV}$ is anti-potential vorticity and $\nabla \cdot \vec{u}$ is divergence). $\mathbf{U}$, $\mathbf{A}$, $\mathbf{T}$ are operators: $\mathbf{U}$ and $\mathbf{A}$ are known operators.

Equations (5) and (6) are for illustration only; the equation that we wish to solve by the GCR method is Eq. (4). In Eq. (4), we know the left hand side, we know the operator, $\mathbf{AU}$, on the right hand side, but we do not know $\vec{v}$.

2. Block-matrix form of Eq. (4)

In matrix form, Eq. (4) is block diagonal,

$$\mathbf{A}\vec{x} = \mathbf{A}\mathbf{U}\vec{v},$$

$$\begin{pmatrix} PV \\ \bar{PV} \\ \nabla \cdot \vec{u} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{PV}\mathbf{U}_s & 0 & 0 \\ 0 & \mathbf{A}_{\bar{PV}}\mathbf{U}_{{}^{u}p} & 0 \\ 0 & 0 & \mathbf{A}_{\nabla \cdot u}\mathbf{U}_{\chi} \end{pmatrix} \begin{pmatrix} s \\ {}^{u}p \\ \chi \end{pmatrix}. \quad (8)$$

Due to the block diagonal form of Eq. (4) we have three uncoupled equations to solve,

$$PV = \mathbf{A}_{PV}\mathbf{U}_s s, \quad (8a)$$

$$\bar{PV} = \mathbf{A}_{\bar{PV}}\mathbf{U}_{{}^{u}p}\,{}^{u}p, \quad (8b)$$

$$\nabla \cdot \vec{u} = \mathbf{A}_{\nabla \cdot u}\mathbf{U}_{\chi}\chi. \quad (8c)$$

$\mathbf{U}_s$, $\mathbf{U}_{{}^{u}p}$, $\mathbf{U}_{\chi}$ are columns of the full $\mathbf{U}$-matrix operator and $\mathbf{A}_{PV}$, $\mathbf{A}_{\bar{PV}}$, $\mathbf{A}_{\nabla \cdot u}$ are rows of the full $\mathbf{A}$-matrix operator.

3. Explicit forms of the equations

3a. The 'balanced rotational' equation - Eq. (8a)

In the bulk of the domain (ie excluding the top-most and bottom-most levels), the PV operator is,

$$PV = \frac{\theta_{0z}}{\rho_0}\zeta_s - \frac{f\theta_{0z}}{\rho_0^2}\left\{\frac{1-\kappa}{R_0\Pi_0\hat{\theta}_0}p_s + \frac{\rho_0}{\hat{\theta}_0}\hat{Q}\right\} + \frac{fg}{\rho_0 c_p}\left\{\frac{1}{\hat{\Pi}_{0z}^2}R - \frac{2\Pi_{0zz}}{\hat{\Pi}_{0z}^3}\hat{S}\right\}, \quad (9)$$

where

$$\zeta_s = \vec{k}\cdot\nabla\times\vec{u}_s, \quad (10)$$

$$Q = \frac{\theta_0}{\Pi_{0z}}\frac{\partial}{\partial z}\left[\kappa\frac{\Pi_0}{p_0}p_s\right], \quad (11)$$

$$R = \frac{\partial^2}{\partial z^2}\left[\kappa\frac{\Pi_0}{p_0}p_s\right], \quad (12)$$

$$S = \frac{\partial}{\partial z}\left[\kappa\frac{\Pi_0}{p_0}p_s\right]. \quad (13)$$

Variables with subscript 0 refer to reference state quantities, except R_0 which is the gas constant (the subscript 0 has been added to distinguish it from R in Eq. (12)) and the hat ˆ denotes vertical interpolation (ie half-to-full or full-to-half levels). Variables with subscript s are balanced increments (ie they are associated with the balanced streamfunction s - see below) and subscript z denotes vertical derivative.

There is no PV calculated at the top and bottom of the domain. Instead, we calculate the PV-like quantities PV_1 and PV_2,

$$PV_1 = \int_{z=0}^{z_{top}} \mathrm{d}z\left(\rho_0\zeta_s - f\left\{\frac{1-\kappa}{R_0\Pi_0\hat{\theta}_0}p_s + \frac{\rho_0}{\hat{\theta}_0}\hat{Q}\right\}\right), \quad (14)$$

$$PV_2 = f\int_{z=0}^{z_{top}} \mathrm{d}z\hat{Q} - f\int_{z=0}^{z_{top}} \mathrm{d}z\frac{\theta_{0z}}{\rho_0}\int_{z'=0}^{z} \mathrm{d}z'\left\{\frac{1-\kappa}{R_0\Pi_0\hat{\theta}_0}p_s + \frac{\rho_0}{\hat{\theta}_0}\hat{Q}\right\} + \int_{z=0}^{z_{top}} \mathrm{d}z\frac{\theta_{0z}}{\rho_0}\left\{\int_{z'=0}^{z} \mathrm{d}z'\rho_0\zeta_s - PV_1\frac{\int_{z'=0}^{z}\mathrm{d}z'\rho_0}{\int_{z'=0}^{z_{top}}\mathrm{d}z'\rho_0}\right\}. \quad (15)$$

The $\mathbf{U}_s$ operator gives the u_s, v_s, p_s fields present in the PV definitions above. The $\mathbf{U}_s$ operator acts on the balanced streamfunction field s,

$$\begin{pmatrix} u_s \\ v_s \\ p_s \end{pmatrix} = \mathbf{U}_s s = \begin{pmatrix} -\partial s/\partial y \\ \partial s/\partial x \\ \nabla^{-2}\nabla\cdot(f\rho_0\nabla s) \end{pmatrix}, \quad (16)$$

where the bottom row of Eq. (16) is the linear balance equation, and the ∇ operators are horizontal only.

3b. The 'unbalanced rotational' equation - Eq. (8b)

The anti-PV operator is the linear balance residual,

$$P\bar{V} = -f\nabla(f\rho_0) \cdot (\vec{k} \times \vec{u}_{^u p}) + f^2\rho_0\zeta_{^u p} - f\nabla^2({}^u p), \quad (17)$$

$$\text{where} \quad \zeta_{^u p} = \vec{k} \cdot \nabla \times \vec{u}_{^u p}. \quad (18)$$

The $\mathbf{U}_{P\bar{V}}$ operator gives the $u_{^u p}$, $v_{^u p}$, ${}^u p$ fields present in the PV definitions above. The $\mathbf{U}_{^u p}$ operator acts on the unbalanced pressure field ${}^u p$. In the bulk, the $u_{^u p}$, $v_{^u p}$, ${}^u p$ are found by setting $PV = 0$ in Eq. (9), and rearranging for $\zeta_{^u p}$,

$$\zeta_{^u p} = \frac{f}{\rho_0}\left\{\frac{1-\kappa}{R\Pi_0\hat{\theta}_0}{}^u p + \frac{\rho_0}{\hat{\theta}_0}\hat{Q}\right\} - \frac{fg}{\theta_{0z}c_p}\left\{\frac{1}{\hat{\Pi}_{0z}^2}R - \frac{2\Pi_{0zz}}{\hat{\Pi}_{0z}^3}\hat{S}\right\}, \quad (19)$$

where Q, R, S are found from Eqs. (11) to (13) but calculated from ${}^u p$ instead of p_s. The values of $\zeta_{^u p}$ at the top and bottom are found by setting $PV_1 = 0$ and $PV_2 = 0$ (not done here). The vorticity is converted to streamfunction,

$$\psi_{^u p} = \nabla^{-2}\zeta_{^u p}, \quad (20)$$

which then gives the rotationally unbalanced velocities,

$$\vec{u}_{^u p} = \vec{k} \times \nabla\psi_{^u p}, \quad (21)$$

giving $u_{^u p}$, $v_{^u p}$ as found in Eqs. (17) and (18).

3b. The divergent equation - Eq. (8c)

This is straightforward and does not require the GCR solver, $\mathbf{A}_{\nabla \cdot u}\mathbf{U}_{\chi} = \nabla^2$ in Eq. (8c).