Approximate 'vertical-only' preconditioning of the PV equations

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1. The balanced transform

For the balanced GCR solver, the following represents a simplified form of the **U**-transform

$$\begin{pmatrix} \zeta' \\ p' \end{pmatrix} = \begin{pmatrix} \nabla^2 \\ f\rho_0 \end{pmatrix} \psi'_B + \begin{pmatrix} 0 \\ \langle p'_B \rangle \end{pmatrix}. \tag{1}$$

This is based in (22) of [1]. ψ'_B is a 3-d field where $\langle \psi'_B \rangle = 0$ ($\langle \cdot \rangle$ is level-by-level global mean) and $\langle p'_B \rangle$ is the level-by-level global mean balanced pressure. All quantities are stored on ψ-points. This includes PV, which has the full form (15) of [1].

$$\begin{aligned} PV &= \frac{\theta_{0z}}{\rho_0}\zeta' - \frac{f\theta_{0z}}{\rho_0^2}\rho' + \frac{f}{\rho_0}\theta'_z, \\ &= \frac{\theta_{0z}}{\rho_0}\zeta' - \frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R\Pi_0\hat{\theta}_0}p' - \frac{f\theta_{0z}}{\rho_0\hat{\theta}_0}\overline{\frac{\theta_0}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0}{p_0}p'\right)} + \\ &\quad \frac{fg}{\rho_0 c_p \hat{\Pi}_{0z}^2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0}{p_0}p'\right) - \frac{2fg\Pi_{0zz}}{\rho_0 c_p \hat{\Pi}_{0z}^3}\overline{\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0}{p_0}p'\right)}, \end{aligned} \tag{2}$$

where the overbar and hats denote vertical interpolation from θ-levels to p-levels, and a subscript '0' indicates a reference state quantity. Inserting (1) into (2) gives

$$\begin{aligned} PV = &\frac{\theta_{0z}}{\rho_0}\nabla^2\psi'_B - \\ &\frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R\Pi_0\hat{\theta}_0}[f\rho_0\psi'_B + \langle p'_B\rangle] - \\ &\frac{f\theta_{0z}}{\rho_0\hat{\theta}_0}\overline{\frac{\theta_0}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0}{p_0}[f\rho_0\psi'_B + \langle p'_B\rangle]\right)} + \\ &\frac{fg}{\rho_0 c_p \hat{\Pi}_{0z}^2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0}{p_0}[f\rho_0\psi'_B + \langle p'_B\rangle]\right) - \\ &\frac{2fg\Pi_{0zz}}{\rho_0 c_p \hat{\Pi}_{0z}^3}\overline{\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0}{p_0}[f\rho_0\psi'_B + \langle p'_B\rangle]\right)}. \end{aligned} \tag{3}$$

This is the PV given entirely in terms of the balanced control variables. For the bulk PV at level k, the discretization is as follows (note that all quantities are at horizontal position i, j unless otherwise stated)

$$\begin{aligned} PV(i,j,k) = &\frac{\theta_{0z}}{r^2\rho_0\cos\phi_j^\psi}\left(\frac{\psi'_B(i+1,j,k) - 2\psi'_B(k) + \psi'_B(i-1,j,k)}{\delta\lambda^2\cos\phi_j^\psi} + \right. \\ &\left.\frac{\cos\phi_{j+1}^p(\psi'_B(i,j+1,k) - \psi'_B(k)) - \cos\phi_j^p(\psi'_B(k) - \psi'_B(i,j-1,k))}{\delta\phi^2}\right) - \\ &\frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R\Pi_0\hat{\theta}_0}[f\rho_0(k)\psi'_B(k) + \langle p'_B\rangle(k)] - \end{aligned}$$

$$\frac{f\kappa}{\rho_0\hat{\Pi}_{0z}}\left(\theta_{0z}+\frac{2g\Pi_{0zz}}{c_p\hat{\Pi}_{0z}^2}\right)\left(\frac{\alpha_1(k)}{r^p(k+1)-r^p(k)}\left[\frac{\Pi_0(k+1)}{p_0(k+1)}[f\rho_0(k+1)\psi'_B(k+1)+\langle p'_B\rangle(k+1)]-\right.\right.$$

$$\left.\frac{\Pi_0(k)}{p_0(k)}[f\rho_0(k)\psi'_B(k)+\langle p'_B\rangle(k)]\right]+$$

$$\frac{\beta_1(k)}{r^p(k)-r^p(k-1)}\left[\frac{\Pi_0(k)}{p_0(k)}[f\rho_0(k)\psi'_B(k)+\langle p'_B\rangle(k)]-\right.$$

$$\left.\left.\frac{\Pi_0(k-1)}{p_0(k-1)}[f\rho_0(k-1)\psi'_B(k-1)+\langle p'_B\rangle(k-1)]\right]\right)+$$

$$\frac{fg\kappa}{\rho_0 c_p\hat{\Pi}_{0z}^2}\frac{1}{r^\theta(k)-r^\theta(k-1)}\times$$

$$\left(\frac{\Pi_0(k+1)[f\rho_0(k+1)\psi'_B(k+1)+\langle p'_B\rangle(k+1)]/p_0(k+1)-\Pi_0(k)[f\rho_0(k)\psi'_B(k)+\langle p'_B\rangle(k)]/p_0(k)}{r^p(k+1)-r^p(k)}-\right.$$

$$\left.\frac{\Pi_0(k)[f\rho_0(k)\psi'_B(k)+\langle p'_B\rangle(k)]/p_0(k)-\Pi_0(k-1)[f\rho_0(k-1)\psi'_B(k-1)+\langle p'_B\rangle(k-1)]/p_0(k-1)}{r^p(k)-r^p(k-1)}\right) \tag{4}$$

In writing Eq. (4), for simplicity:

- Horizontal interpolation of reference state quantities to ψ-points is ignored for preconditioning. Each value is taken at its 'home' point that has the same horizontal index as the ψ-point.
- Some quantities that are part of a compound vertical interpolation can be approximately 'removed' from the compound and cancel with individual terms outside. For example (note that both overbar and hat indicate vertical interpolation)

$$\frac{f^2\theta_{0z}}{\rho_0\hat{\theta}_0}\overline{\frac{\theta_0}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0\rho_0}{p_0}\psi'_B\right)}\approx\frac{f^2\theta_{0z}}{\rho_0}\overline{\frac{1}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0\rho_0}{p_0}\psi'_B\right)}.$$

$\alpha_1(k)$ and $\beta_1(k)$ are vertical interpolation coefficients,

$$\alpha_1(k)=\frac{r^p(k)-r^\theta(k-1)}{r^\theta(k)-r^\theta(k-1)}, \tag{5}$$

$$\beta_1(k)=\frac{r^\theta(k)-r^p(k)}{r^\theta(k)-r^\theta(k-1)}. \tag{6}$$

1.1 Diagonal preconditioning of the balanced equation

Diagonal preconditioning involves ignoring terms in the right-hand-side of (4) that are different from position (i, j, k). This gives

$$PV(i,j,k)\approx-\frac{\theta_{0z}}{r^2\rho_0\cos\phi_j^\psi}\left(\frac{2}{\delta\lambda^2\cos\phi_j^\psi}+\frac{\cos\phi_{j+1}^p+\cos\phi_j^p}{\delta\phi^2}\right)\psi'_B(k)-$$

$$\frac{f\theta_{0z}(1-\kappa)}{\rho_0^2R\Pi_0\hat{\theta}_0}[f\rho_0(k)\psi'_B(k)+\langle p'_B\rangle(k)]-$$

$$\frac{f\kappa}{\rho_0\hat{\Pi}_{0z}}\left(\theta_{0z}+\frac{2g\Pi_{0zz}}{c_p\hat{\Pi}_{0z}^2}\right)\left(\frac{A\alpha_1(k)}{r^p(k+1)-r^p(k)}+\frac{B\beta_1(k)}{r^p(k)-r^p(k-1)}\right)\frac{\Pi_0(k)}{p_0(k)}[f\rho_0(k)\psi'_B(k)+\langle p'_B\rangle(k)]-$$

$$\frac{fg\kappa}{\rho_0 c_p \hat{\Pi}_{0z}^2}\frac{1}{r^\theta(k)-r^\theta(k-1)}\times$$

$$\left(\frac{A}{r^p(k+1)-r^p(k)}+\frac{B}{r^p(k)-r^p(k-1)}\right)\frac{\Pi_0(k)}{p_0(k)}\left[f\rho_0(k)\psi'_B(k)+\langle p'_B\rangle(k)\right]. \tag{7}$$

The extra factors A and B that appear in (7) are unity except in the following circumstances:

- at the top of the domain ($k = N$), $A = 0$ and
- at the bottom of the domain ($k = 1$), $B = 0$.

This is an application of the Neumann boundary conditions. Equation (7) can be parametrised in the following way (at each horizontal position)

$$PV(k) \approx (\lambda + \mu f\rho_0)\psi'_B(k) + \mu\langle p'_B\rangle(k), \tag{8}$$

where, from (7)

$$\lambda = -\frac{\theta_{0z}}{r^2\rho_0\cos\phi_j^\psi}\left(\frac{2}{\delta\lambda^2\cos\phi_j^\psi}+\frac{\cos\phi_{j+1}^p+\cos\phi_j^p}{\delta\phi^2}\right), \tag{9}$$

$$\mu = -\frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R\Pi_0\hat{\theta}_0}-\frac{f\kappa}{\rho_0\hat{\Pi}_{0z}}\left(\theta_{0z}+\frac{2g\Pi_{0zz}}{c_p\hat{\Pi}_{0z}^2}\right)\left(\frac{A\alpha_1(k)}{r^p(k+1)-r^p(k)}+\frac{B\beta_1(k)}{r^p(k)-r^p(k-1)}\right)\frac{\Pi_0(k)}{p_0(k)}-$$
$$\frac{fg\kappa}{\rho_0 c_p\hat{\Pi}_{0z}^2}\frac{1}{r^\theta(k)-r^\theta(k-1)}\left(\frac{A}{r^p(k+1)-r^p(k)}+\frac{B}{r^p(k)-r^p(k-1)}\right)\frac{\Pi_0(k)}{p_0(k)}. \tag{10}$$

The unknowns $\psi'_B(k)$ and $\langle p'_B\rangle\,(k)$ are determined (noting the $\langle\psi'_B(k)\rangle = 0$ requirement) by multiplying (8) by $1/(\lambda + \mu f\rho_0)$ and then taking the global mean

$$\langle\frac{1}{\lambda+\mu f\rho_0}PV(k)\rangle = \langle\psi'_B(k)\rangle + \langle\frac{\mu}{\lambda+\mu f\rho_0}\rangle\langle p'_B\rangle(k),$$

$$\langle p'_B\rangle(k) = \left(\langle\frac{\mu}{\lambda+\mu f\rho_0}\rangle\right)^{-1}\langle\frac{1}{\lambda+\mu f\rho_0}PV(k)\rangle, \tag{11}$$

$$\psi'_B(k) = \frac{PV(k)-\mu\langle p'_B\rangle(k)}{\lambda+\mu f\rho_0}. \tag{12}$$

1.2 Vertical preconditioning of the balanced equation

Vertical preconditioning involves adding-up vertical terms from neighbouring points of (4) in the horizontal. Doing this removes the horizontal derivatives. Only the vertical terms remain

$$PV(k) \approx \mu(k)\omega(k) + \mu^+(k)\omega(k+1) + \mu^-(k)\omega(k-1), \tag{13}$$

where

$$\mu(k) = -\frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R\Pi_0\hat{\theta}_0}+\frac{f\kappa}{\rho_0\hat{\Pi}_{0z}}\left(\theta_{0z}+\frac{2g\Pi_{0zz}}{c_p\hat{\Pi}_{0z}^2}\right)\left(\frac{A\alpha_1(k)}{r^p(k+1)-r^p(k)}-\frac{B\beta_1(k)}{r^p(k)-r^p(k-1)}\right)\frac{\Pi_0(k)}{p_0(k)}-$$
$$\frac{fg\kappa}{\rho_0 c_p\hat{\Pi}_{0z}^2}\frac{1}{r^\theta(k)-r^\theta(k-1)}\frac{\Pi_0(k)}{p_0(k)}\left(\frac{A}{r^p(k+1)-r^p(k)}+\frac{B}{r^p(k)-r^p(k-1)}\right), \tag{14}$$

$$\mu^+(k) = -\frac{f\kappa}{\rho_0\hat{\Pi}_{0z}}\left(\theta_{0z}+\frac{2g\Pi_{0zz}}{c_p\hat{\Pi}_{0z}^2}\right)\frac{A\alpha_1(k)}{r^p(k+1)-r^p(k)}\frac{\Pi_0(k+1)}{p_0(k+1)}+$$
$$\frac{fg\kappa}{\rho_0 c_p\hat{\Pi}_{0z}^2}\frac{1}{r^\theta(k)-r^\theta(k-1)}\frac{\Pi_0(k+1)}{p_0(k+1)}\frac{A}{r^p(k+1)-r^p(k)}, \tag{15}$$

$$\mu^{-}(k) = \frac{f\kappa}{\rho_0 \hat{\Pi}_{0z}} \left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p \hat{\Pi}_{0z}^2} \right) \frac{B\beta_1(k)}{r^p(k) - r^p(k-1)} \frac{\Pi_0(k-1)}{p_0(k-1)} + \frac{fg\kappa}{\rho_0 c_p \hat{\Pi}_{0z}^2} \frac{1}{r^\theta(k) - r^\theta(k-1)} \frac{\Pi_0(k-1)}{p_0(k-1)} \frac{B}{r^p(k) - r^p(k-1)}, \tag{16}$$

$$\omega(k) = f\rho_0(k)\psi'_B(k) + \langle p'_B \rangle(k). \tag{17}$$

Equation (13) is a tridiagonal system of equations which can be solved for $\omega(k)$ at each horizontal position. Once $\omega(k)$ is known, it is then possible to derive the required fields from (17)

$$\langle p'_B \rangle(k) = \langle f\rho_0(k) \rangle \langle \frac{\omega(k)}{f\rho_0(k)} \rangle, \tag{18}$$

$$\psi'_B(k) = \frac{\omega(k) - \langle p'_B \rangle(k)}{f\rho_0(k)}. \tag{19}$$

2. The unbalanced transform

For the unbalanced GCR solver, the following represents a form of the **U**-transform,

$$\begin{pmatrix} \zeta' \\ p' \end{pmatrix} = \begin{pmatrix} \frac{f(1-\kappa)}{\rho_0 R \Pi_0 \hat{\theta}_0} + \frac{f}{\hat{\theta}_0} \overline{\frac{\theta_0}{\Pi_{0z}} \frac{\partial}{\partial z}\left(\kappa \frac{\Pi_0}{p_0} \cdot\right)} - \frac{fg}{c_p \theta_{0z} \hat{\Pi}_{0z}^2} \frac{\partial^2}{\partial z^2}\left(\frac{\kappa \Pi_0}{p_0} \cdot\right) + \frac{2fg\Pi_{0zz}}{c_p \theta_{0z} \hat{\Pi}_{0z}^3} \overline{\frac{\partial}{\partial z}\left(\frac{\kappa \Pi_0}{p_0} \cdot\right)} \\ 1 \end{pmatrix} p'_u, \tag{20}$$

where the overbar and hats denote vertical interpolation from θ-levels to p-levels. This is taken from (26) of [1]. All quantities are stored on p-points. This includes $\overline{PV}$, which has the simplified form,

$$\overline{PV} = f\rho_0 \zeta' - \nabla^2 p', \tag{21}$$

(see (25) of [1] and the overbar on $\overline{PV}$ indicates anti-PV, not vertical interpolation). Inserting (20) into (21) gives,

$$\overline{PV} = \frac{f^2(1-\kappa)}{R\Pi_0 \hat{\theta}_0} p'_u + \frac{f^2 \rho_0}{\hat{\theta}_0} \overline{\frac{\theta_0}{\Pi_{0z}} \frac{\partial}{\partial z}\left(\kappa \frac{\Pi_0}{p_0} p'_u\right)} - \frac{f^2 g \rho_0}{c_p \theta_{0z} \hat{\Pi}_{0z}^2} \frac{\partial^2}{\partial z^2}\left(\frac{\kappa \Pi_0}{p_0} p'_u\right) + \frac{2f^2 g \Pi_{0zz} \rho_0}{c_p \theta_{0z} \hat{\Pi}_{0z}^3} \overline{\frac{\partial}{\partial z}\left(\frac{\kappa \Pi_0}{p_0} p'_u\right)} - \nabla^2 p'_u. \tag{22}$$

This is the $\overline{PV}$ entirely in terms of the unbalanced pressure control variable. For the bulk $\overline{PV}$ at level k, the discretization is as follows (note that all quantities are at horizontal position i, j unless otherwise stated),

$$\overline{PV}(i,j,k) = \frac{f^2(1-\kappa)}{R\Pi_0 \hat{\theta}_0} p'_u(k) + \frac{f^2 \kappa \rho_0}{\hat{\Pi}_{0z}} \left(1 + \frac{2g\Pi_{0zz}}{c_p \theta_{0z} \hat{\Pi}_{0z}^2}\right) \left(\frac{\alpha_1(k)}{r^p(k+1) - r^p(k)} \left[\frac{\Pi_0(k+1)}{p_0(k+1)} p'_u(k+1) - \frac{\Pi_0(k)}{p_0(k)} p'_u(k) \right] + \frac{\beta_1(k)}{r^p(k) - r^p(k-1)} \left[\frac{\Pi_0(k)}{p_0(k)} p'_u(k) - \frac{\Pi_0(k-1)}{p_0(k-1)} p'_u(k-1) \right] \right) - \frac{f^2 g \kappa \rho_0}{c_p \theta_{0z} \hat{\Pi}_{0z}^2} \frac{1}{(r^\theta(k) - r^\theta(k-1))} \left(\frac{\Pi_0(k+1) p'_u(k+1)/p_0(k+1) - \Pi_0(k) p'_u(k)/p_0(k)}{r^p(k+1) - r^p(k)} - \right.$$

$$\left.\frac{\Pi_0(k)p'_u(k)/p_0(k)-\Pi_0(k-1)p'_u(k-1)/p_0(k-1)}{r^p(k)-r^p(k-1)}\right)-$$

$$\frac{1}{r^2\cos\phi_j^p}\left(\frac{p'_u(i+1,j,k)-2p'_u(k)+p'_u(i-1,j,k)}{\delta\lambda^2\cos\phi_j^p}+\right.$$

$$\left.\frac{\cos\phi_j^\psi(p'_u(i,j+1,k)-p'_u(k))-\cos\phi_{j-1}^\psi(p'_u(k)-p'_u(i,j-1,k))}{\delta\phi^2}\right). \quad (23)$$

In writing (23), the same approximations are used as for (4) - see bullet points after (4). $\alpha_1(k)$ and $\beta(k)$ are vertical interpolation coefficients given as (5) and (6).

Vertical preconditioning involves adding-up vertical terms from neighbouring points of (23) in the horizontal. Doing this removes the horizontal derivatives. Only the vertical terms remain,

$$\overline{PV}(i,j,k)=\frac{f^2(1-\kappa)}{R\Pi_0\hat{\theta}_0}p'_u(k)+$$

$$\frac{f^2\kappa\rho_0}{\hat{\Pi}_{0z}}\left(1+\frac{2g\Pi_{0zz}}{c_p\theta_{0z}\hat{\Pi}_{0z}^2}\right)\left(\overset{A}{\boxed{\frac{\alpha_1(k)}{r^p(k+1)-r^p(k)}\left[\frac{\Pi_0(k+1)}{p_0(k+1)}p'_u(k+1)-\frac{\Pi_0(k)}{p_0(k)}p'_u(k)\right]}}+\right.$$

$$\left.\overset{B}{\boxed{\frac{\beta_1(k)}{r^p(k)-r^p(k-1)}\left[\frac{\Pi_0(k)}{p_0(k)}p'_u(k)-\frac{\Pi_0(k-1)}{p_0(k-1)}p'_u(k-1)\right]}}\right)-$$

$$\frac{f^2g\kappa\rho_0}{c_p\theta_{0z}\hat{\Pi}_{0z}^2}\frac{1}{(r^\theta(k)-r^\theta(k-1))}\left(\overset{A}{\boxed{\frac{\Pi_0(k+1)p'_u(k+1)/p_0(k+1)-\Pi_0(k)p'_u(k)/p_0(k)}{r^p(k+1)-r^p(k)}}}-\right.$$

$$\left.\overset{B}{\boxed{\frac{\Pi_0(k)p'_u(k)/p_0(k)-\Pi_0(k-1)p'_u(k-1)/p_0(k-1)}{r^p(k)-r^p(k-1)}}}\right). \quad (24)$$

Boxed terms need attention at the vertical boundaries. For Neumann boundary conditions ($\theta = 0$ (increments) at the top and bottom), terms marked 'A' are zero when $k = N$ and terms marked 'B' are zero when $k = 1$.

Given $\overline{PV}$, Eq. (24) is inverted for p'_u with a tridiagonal solver for the preconditioning step.

Reference

[1] Bannister R.N., Cullen M.J.P., 2008, New PV-based variables for Met Office VAR, Version 07.