Derivation of non-linear and linear Defant model

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Here we show the detailed derivation of the non-linear and linear Defant model, following closely the methods in [1].

1 Derivation of non-linear Defant equations

1.1 Conservation of momentum equations

The momentum equation is

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v}.\nabla\mathbf{v} - \alpha\nabla p - g\mathbf{k} + f \times \mathbf{v} \tag{1}$$

where $mbfv(x, y, z, t) = (u(x, t), v(y, t), w(z, t))$ is velocity vector in x, y, z directions and time, t, α is specific density, p is pressure, g is gravity, and f is normalised Coriolis parameter.
We can rewrite non-linear equation (1) for each coordinate as follows,

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} - \alpha\frac{\partial p}{\partial x} + fv \tag{2}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - w\frac{\partial v}{\partial z} - fu \tag{3}$$

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} - \alpha\frac{\partial p}{\partial z} - g \tag{4}$$

where we have assumed the system to be homogeneous, i.e. $\frac{\partial}{\partial y} = 0$, have neglected the Coriolis force in vertical direction and ignored gravity term g in horizontal variables.
Since the large scale motion is in hydrostatic balance, i.e. $\frac{\partial p_0}{\partial z} = -\frac{g}{\alpha_0}$, where we have defined $p = p_0 + p'$ and $\alpha = \alpha_0 + \alpha'$ with synoptic scale terms having subscripts $_0$ and mesoscale terms having superscripts $'$, we can rewrite the last two terms in the equation (4) as follows

$$\begin{aligned}(\alpha_0 + \alpha')\frac{\partial}{\partial z}(p_0 + p') + g &= \alpha_0\left(1 + \frac{\alpha'}{\alpha_0}\right)\frac{\partial p'}{\partial z} - g + g - \frac{\alpha'}{\alpha_0}g \\ &\approx \alpha_0\frac{\partial p'}{\partial z} - \frac{\alpha'}{\alpha_0}g\end{aligned}$$

where we also have assumed that $\frac{|\alpha'|}{\alpha_0} << 1$.
Moreover, we can rewrite the term $\frac{\alpha'}{\alpha_0}g$ by first using the relation $\theta = T(1000/p)^{R_d/c_p}$ in the ideal gas law, $\alpha = \frac{RT}{p}$, and then logarithmically differentiating the it. Hence,

$$\alpha = R\theta 1000^{-\frac{R_d}{c_p}} p^{-\frac{c_v}{c_p}}$$

where we have used that $c_p = c_v + R_d$. This can be rewritten as

$$\ln \alpha = \ln R + \ln \theta - \frac{R_d}{c_p} \ln 1000 - \frac{c_v}{c_p} \ln p$$

which when differentiated, becomes

$$\frac{d\alpha}{\alpha} = \frac{d\theta}{\theta} - \frac{c_v}{c_p} \frac{dp}{p}.$$

Assuming that the changes of α, θ, p are much less than their absolute magnitudes, i.e.

$$|d\alpha| << \alpha, |d\theta| << \theta, |dp| << p$$

and if $\alpha \approx \alpha_0$, $\theta \approx \theta_0$, and $p \approx p_0$ then we can approximate above by

$$\frac{\alpha'}{\alpha_0} \approx \frac{\theta'}{\theta_0} - \frac{c_v}{c_p} \frac{p'}{p_0}. \tag{5}$$

Equation (5) can be simplified when the vertical scale of the circulation, L_z, is much smaller than the scale depth of the atmosphere, H_α. Using the scale analysis we have

$$\left| \alpha_0 \frac{\partial p'}{\partial z} \right| \approx \alpha_0 \frac{|p'|}{L_z},$$

and assuming that the vertical mesoscale pressure perturbation and the density perturbation terms are of the same order of magnitude, i.e. the last two terms in equation (4) can be approximately equated

$$\left| \alpha_0 \frac{\partial p'}{\partial z} \right| \approx \left| \frac{\alpha'}{\alpha_0} \right| g$$

and using the ideal gas law we have that

$$\frac{RT_0 |p'|}{L_z p_0} \approx \frac{|\alpha'|}{\alpha_0} g.$$

Rewriting, above yields

$$\frac{|p'|}{p_0} \approx \frac{L_z g}{RT_0} \frac{|\alpha'|}{\alpha_0} = \frac{L_z g}{p_0 \alpha_0} \frac{|\alpha'|}{\alpha_0} = \frac{L_z}{D} \frac{|\alpha'|}{\alpha_0} \approx \frac{L_z}{H_\alpha} \frac{|\alpha'|}{\alpha_0}$$

where $H_\alpha \approx D \approx \frac{p_0}{\rho_0 g}$. Thus, if $L_z << H_\alpha$, then

$$\frac{|p'|}{p_0} << \frac{|\alpha'|}{\alpha_0} \text{ and } \frac{|\alpha'|}{\alpha_0} \approx \frac{|\theta'|}{\theta_0}.$$

Hence, the vertical component of acceleration can be written as,

$$\frac{\partial w}{\partial t} = -u \frac{\partial w}{\partial x} - w \frac{\partial w}{\partial z} - \alpha_0 \frac{\partial p'}{\partial z} + \frac{\theta'}{\theta_0} g. \tag{6}$$

Note that equation (6) is a shallow form of the vertical motion equation. If deep atmospheric circulations are needed, then use

$$\frac{|\alpha'|}{\alpha_0} = \frac{\theta'}{\theta_0} - \frac{c_v p'}{c_p p_0}.$$

1.2 Conservation of mass

Conservation of mass is given by

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho v}{\partial y} - \frac{\partial \rho w}{\partial z} \tag{7}$$

$$= -\nabla.\rho\mathbf{v}. \tag{8}$$

If considering homogeneous problem as for conservation of momentum, then equation (7) simplifies to

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial z}. \tag{9}$$

1.3 Conservation of heat

Conservation of heat is given by

$$\frac{\partial \theta}{\partial t} = -u\frac{\partial \theta}{\partial x} - v\frac{\partial \theta}{\partial y} - w\frac{\partial \theta}{\partial z} \tag{10}$$

$$= -\mathbf{v}.\nabla\theta \tag{11}$$

where we have ignored the source and sink term, which includes the sum of processes such as freezing, melting, condensation, evaporation, chemical reactions, etc.

Again, considering homogeneous problem, then equation (10) simplifies to

$$\frac{\partial \theta}{\partial t} = -u\frac{\partial \theta}{\partial x} - w\frac{\partial \theta}{\partial z}. \tag{12}$$

1.4 Summary of non-linear equations

Thus the non-linear conservation law equations, assuming homogeneity and that $\frac{|\alpha'|}{\alpha_0} << 1$ are

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} - \alpha\frac{\partial p}{\partial x} + fv \tag{13}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - w\frac{\partial v}{\partial z} - fu \tag{14}$$

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} - \alpha_0\frac{\partial p'}{\partial z} + \frac{\theta'}{\theta_0}g \tag{15}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial z} \tag{16}$$

$$\frac{\partial \theta}{\partial t} = -u\frac{\partial \theta}{\partial x} - w\frac{\partial \theta}{\partial z}. \tag{17}$$

2 Anelastic non-linear model

The above model, given by equations (13)-(17), becomes an anelastic model when

$$\frac{\partial \rho}{\partial t} = 0.$$

Hence, the model is

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} - \alpha\frac{\partial p}{\partial x} + fv && (18)\\
\frac{\partial v}{\partial t} &= -u\frac{\partial v}{\partial x} - w\frac{\partial v}{\partial z} - fu && (19)\\
\frac{\partial w}{\partial t} &= -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} - \alpha\frac{\partial p}{\partial z} + \frac{\theta'}{\theta_0}g && (20)\\
0 &= \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} && (21)\\
\frac{\partial \theta}{\partial t} &= -u\frac{\partial \theta}{\partial x} - w\frac{\partial \theta}{\partial z}. && (22)
\end{aligned}$$

The unknowns in this system are u, v, w, p, θ, the equations (18)-(20) advance the momentum variables in time, however, we require to know the pressure term to step them forwards. For this we use the shallow water continuity equation (21), which we differentiate with respect to time,

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z}\right) && (23)\\
&= \frac{\partial}{\partial x}\left(\rho\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial w}{\partial t}\right). && (24)
\end{aligned}$$

Now substituting equations (18) and (20) into (24) gives

$$\begin{aligned}
0 &= \frac{\partial}{\partial x}\left(-\rho u\frac{\partial u}{\partial x} - \rho w\frac{\partial u}{\partial z} - \frac{\partial p}{\partial x} + \rho fv\right) + \frac{\partial}{\partial z}\left(-\rho u\frac{\partial w}{\partial x} - \rho w\frac{\partial w}{\partial z} - \frac{\partial p}{\partial z} + \rho\frac{\theta'}{\theta_0}g\right) && (25)\\
\therefore \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} &= \frac{\partial}{\partial x}\left(-\rho u\frac{\partial u}{\partial x} - \rho w\frac{\partial u}{\partial z} + \rho fv\right) + \frac{\partial}{\partial z}\left(-\rho u\frac{\partial w}{\partial x} - \rho w\frac{\partial w}{\partial z} + \rho\frac{\theta'}{\theta_0}g\right) && (26)\\
\therefore \nabla^2_{x,z}p &= \frac{\partial}{\partial x}\left(-\rho u\frac{\partial u}{\partial x} - \rho w\frac{\partial u}{\partial z} + \rho fv\right) + \frac{\partial}{\partial z}\left(-\rho u\frac{\partial w}{\partial x} - \rho w\frac{\partial w}{\partial z} + \rho\frac{\theta'}{\theta_0}g\right) && (27)
\end{aligned}$$

References

[1] R. A. Pielke, *Mesoscale Meteorological Modeling*, 2nd edition, Academic press, 2002