Approximate vertical-only tridiagonal preconditioning of PV and anti-PV equations

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1. The balanced transform

For the balanced GCR solver, the following represents a simplified form of the **U**-transform,

$$\begin{pmatrix} \zeta' \\ p' \end{pmatrix} = \begin{pmatrix} \nabla^2 \\ f\rho_0 \end{pmatrix} \psi'_B. \tag{1}$$

All quantities are stored on ψ-points. This includes PV, which has the full form,

$$\begin{aligned} PV &= \frac{\theta_{0z}}{\rho_0}\zeta' - \frac{f\theta_{0z}}{\rho_0^2}\rho' + \frac{f}{\rho_0}\theta'_z, \\ &= \frac{\theta_{0z}}{\rho_0}\zeta' - \frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R\Pi_0\hat{\theta}_0}p' - \frac{f\theta_{0z}}{\rho_0\hat{\theta}_0}\overline{\frac{\theta_0}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0}{p_0}p'\right)} + \\ &\quad \frac{fg}{\rho_0 c_p\hat{\Pi}_{0z}^2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0}{p_0}p'\right) - \frac{2fg\Pi_{0zz}}{\rho_0 c_p\hat{\Pi}_{0z}^3}\overline{\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0}{p_0}p'\right)}, \end{aligned} \tag{2}$$

where the overbar and hats denote vertical interpolation from θ-levels to p-levels, and a subscript '0' indicates a reference state quantity. Inserting (1) into (2) gives,

$$\begin{aligned} PV &= \frac{\theta_{0z}}{\rho_0}\nabla^2\psi'_B - \frac{f^2\theta_{0z}(1-\kappa)}{\rho_0 R\Pi_0\hat{\theta}_0}\psi'_B - \frac{f^2\theta_{0z}}{\rho_0\hat{\theta}_0}\overline{\frac{\theta_0}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0\rho_0}{p_0}\psi'_B\right)} + \\ &\quad \frac{f^2 g}{\rho_0 c_p\hat{\Pi}_{0z}^2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0\rho_0}{p_0}\psi'_B\right) - \frac{2f^2 g\Pi_{0zz}}{\rho_0 c_p\hat{\Pi}_{0z}^3}\overline{\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0\rho_0}{p_0}\psi'_B\right)}. \end{aligned} \tag{3}$$

This is the PV given entirely in terms of the balanced streamfunction control variable. For the bulk PV at level k, the discretization is as follows (note that all quantities are at horizontal position i, j unless otherwise stated),

$$\begin{aligned} PV(i,j,k) = &\frac{\theta_{0z}}{r^2\rho_0\cos\phi_j^\psi}\left(\frac{\psi'_B(i+1,j,k) - 2\psi'_B(k) + \psi'_B(i-1,j,k)}{\delta\lambda^2\cos\phi_j^\psi} + \right. \\ &\left. \frac{\cos\phi_{j+1}^p(\psi'_B(i,j+1,k) - \psi'_B(k)) - \cos\phi_j^p(\psi'_B(k) - \psi'_B(i,j-1,k))}{\delta\phi^2}\right) - \\ &\frac{f^2\theta_{0z}(1-\kappa)}{\rho_0 R\Pi_0\hat{\theta}_0}\psi'_B(k) - \\ &\frac{f^2\kappa}{\rho_0\hat{\Pi}_{0z}}\left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p\hat{\Pi}_{0z}^2}\right)\left(\frac{\alpha_1(k)}{r^p(k+1) - r^p(k)}\left[\frac{\Pi_0(k+1)\rho_0(k+1)}{p_0(k+1)}\psi'_B(k+1) - \frac{\Pi_0(k)\rho_0(k)}{p_0(k)}\psi'_B(k)\right] + \right. \\ &\left. \frac{\beta_1(k)}{r^p(k) - r^p(k-1)}\left[\frac{\Pi_0(k)\rho_0(k)}{p_0(k)}\psi'_B(k) - \frac{\Pi_0(k-1)\rho_0(k-1)}{p_0(k-1)}\psi'_B(k-1)\right]\right) + \\ &\frac{f^2 g\kappa}{\rho_0 c_p\hat{\Pi}_{0z}^2}\frac{1}{r^\theta(k) - r^\theta(k-1)}\left(\frac{\Pi_0(k+1)\rho_0(k+1)\psi'_B(k+1)/p_0(k+1) - \Pi_0(k)\rho_0(k)\psi'_B(k)/p_0(k)}{r^p(k+1) - r^p(k)} - \right. \\ &\left. \frac{\Pi_0(k)\rho_0(k)\psi'_B(k)/p_0(k) - \Pi_0(k-1)\rho_0(k-1)\psi'_B(k-1)/p_0(k-1)}{r^p(k) - r^p(k-1)}\right). \end{aligned} \tag{4}$$

In writing Eq. (4), for simplicity:

- Horizontal interpolation of reference state quantities to ψ-points is ignored for preconditioning. Each value is taken at its 'home' point that has the same horizontal index as the ψ-point.
- Some quantities that are part of a compound vertical interpolation can be approximately 'removed' from the compound and cancel with individual terms outside. For example (note that both overbar and hat indicate vertical interpolation),

$$\frac{f^2\theta_{0z}}{\rho_0\hat{\theta}_0}\overline{\frac{\theta_0}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0\rho_0}{p_0}\psi'_B\right)} \approx \frac{f^2\theta_{0z}}{\rho_0}\overline{\frac{1}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0\rho_0}{p_0}\psi'_B\right)}.$$

$\alpha_1(k)$ and $\beta_1(k)$ are vertical interpolation coefficients,

$$\alpha_1(k) = \frac{r^p(k) - r^\theta(k-1)}{r^\theta(k) - r^\theta(k-1)}, \tag{5}$$

$$\beta_1(k) = \frac{r^\theta(k) - r^p(k)}{r^\theta(k) - r^\theta(k-1)}. \tag{6}$$

Vertical preconditioning involves adding-up vertical terms from neighbouring points of Eq. (4) in the horizontal. Doing this removes the horizontal derivatives. Only the vertical terms remain,

$$\begin{aligned}
&PV(i,j,k) \approx \\
&-\frac{f^2\theta_{0z}(1-\kappa)}{\rho_0 R\Pi_0\hat{\theta}_0}\psi'_B(k) - \\
&\frac{f^2\kappa}{\rho_0\hat{\Pi}_{0z}}\left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p\hat{\Pi}_{0z}^2}\right)\left(\boxed{\frac{\alpha_1(k)}{r^p(k+1)-r^p(k)}\left[\frac{\Pi_0(k+1)\rho_0(k+1)}{p_0(k+1)}\psi'_B(k+1) - \frac{\Pi_0(k)\rho_0(k)}{p_0(k)}\psi'_B(k)\right]}^{\,A} + \right.\\
&\left.\boxed{\frac{\beta_1(k)}{r^p(k)-r^p(k-1)}\left[\frac{\Pi_0(k)\rho_0(k)}{p_0(k)}\psi'_B(k) - \frac{\Pi_0(k-1)\rho_0(k-1)}{p_0(k-1)}\psi'_B(k-1)\right]}^{\,B}\right) + \\
&\frac{f^2 g\kappa}{\hat{\rho}_0 c_p\Pi_{0z}^2}\frac{1}{r^\theta(k)-r^\theta(k-1)}\left(\boxed{\frac{\Pi_0(k+1)\rho_0(k+1)\psi'_B(k+1)/p_0(k+1) - \Pi_0(k)\rho_0(k)\psi'_B(k)/p_0(k)}{r^p(k+1)-r^p(k)}}^{\,A} - \right.\\
&\left.\boxed{\frac{\Pi_0(k)\rho_0(k)\psi'_B(k)/p_0(k) - \Pi_0(k-1)\rho_0(k-1)\psi'_B(k-1)/p_0(k-1)}{r^p(k)-r^p(k-1)}}^{\,B}\right).
\end{aligned} \tag{7}$$

Boxed terms need attention at the vertical boundaries. For Neumann boundary conditions ($\theta = 0$ (increments) at the top and bottom), terms marked 'A' are zero when $k = N$ and terms marked 'B' are zero when $k = 1$.

Given PV, Eq. (7) is inverted for ψ'_B with a tridiagonal solver for the preconditioning step.

2. The unbalanced transform

For the unbalanced GCR solver, the following represents a form of the **U**-transform,

$$\begin{pmatrix} \zeta' \\ p' \end{pmatrix} = \begin{pmatrix} \frac{f(1-\kappa)}{\rho_0 R \Pi_0 \hat{\theta}_0} + \frac{f}{\hat{\theta}_0} \overline{\frac{\theta_0}{\Pi_{0z}} \frac{\partial}{\partial z}\left(\kappa \frac{\Pi_0}{p_0} \cdot\right)} - \frac{fg}{c_p \theta_{0z} \hat{\Pi}_{0z}^2} \frac{\partial^2}{\partial z^2}\left(\frac{\kappa \Pi_0}{p_0} \cdot\right) + \frac{2fg\Pi_{0zz}}{c_p \theta_{0z} \hat{\Pi}_{0z}^3} \overline{\frac{\partial}{\partial z}\left(\frac{\kappa \Pi_0}{p_0} \cdot\right)} \\ 1 \end{pmatrix} p'_u, \quad (8)$$

where the overbar and hats denote vertical interpolation from θ-levels to p-levels. All quantities are stored on p-points. This includes $\overline{PV}$, which has the simplified form,

$$\overline{PV} = f\rho_0 \zeta' - \nabla^2 p', \quad (9)$$

(the overbar on $\overline{PV}$ indicates anti-PV, not vertical interpolation). Inserting (8) into (9) gives,

$$\overline{PV} = \frac{f^2(1-\kappa)}{R\Pi_0\hat{\theta}_0} p'_u + \frac{f^2\rho_0}{\hat{\theta}_0} \overline{\frac{\theta_0}{\Pi_{0z}} \frac{\partial}{\partial z}\left(\kappa \frac{\Pi_0}{p_0} p'_u\right)} - \frac{f^2 g \rho_0}{c_p \theta_{0z} \hat{\Pi}_{0z}^2} \frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0}{p_0} p'_u\right) + \frac{2f^2 g \Pi_{0zz} \rho_0}{c_p \theta_{0z} \hat{\Pi}_{0z}^3} \overline{\frac{\partial}{\partial z}\left(\frac{\kappa \Pi_0}{p_0} p'_u\right)} - \nabla^2 p'_u. \quad (10)$$

This is the $\overline{PV}$ entirely in terms of the unbalanced pressure control variable. For the bulk $\overline{PV}$ at level k, the discretization is as follows (note that all quantities are at horizontal position i, j unless otherwise stated),

$$\begin{aligned} \overline{PV}(i,j,k) = {} & \frac{f^2(1-\kappa)}{R\Pi_0\hat{\theta}_0} p'_u(k) + \\ & \frac{f^2 \kappa \rho_0}{\hat{\Pi}_{0z}} \left(1 + \frac{2g\Pi_{0zz}}{c_p \theta_{0z} \hat{\Pi}_{0z}^2}\right) \left(\frac{\alpha_1(k)}{r^p(k+1) - r^p(k)} \left[\frac{\Pi_0(k+1)}{p_0(k+1)} p'_u(k+1) - \frac{\Pi_0(k)}{p_0(k)} p'_u(k) \right] + \right. \\ & \left. \frac{\beta_1(k)}{r^p(k) - r^p(k-1)} \left[\frac{\Pi_0(k)}{p_0(k)} p'_u(k) - \frac{\Pi_0(k-1)}{p_0(k-1)} p'_u(k-1) \right] \right) - \\ & \frac{f^2 g \kappa \rho_0}{c_p \theta_{0z} \hat{\Pi}_{0z}^2} \frac{1}{(r^\theta(k) - r^\theta(k-1))} \left(\frac{\Pi_0(k+1) p'_u(k+1)/p_0(k+1) - \Pi_0(k) p'_u(k)/p_0(k)}{r^p(k+1) - r^p(k)} - \right. \\ & \left. \frac{\Pi_0(k) p'_u(k)/p_0(k) - \Pi_0(k-1) p'_u(k-1)/p_0(k-1)}{r^p(k) - r^p(k-1)} \right) - \\ & \frac{1}{r^2 \cos\phi_j^p} \left(\frac{p'_u(i+1,j,k) - 2p'_u(k) + p'_u(i-1,j,k)}{\delta\lambda^2 \cos\phi_j^p} + \right. \\ & \left. \frac{\cos\phi_j^\psi (p'_u(i,j+1,k) - p'_u(k)) - \cos\phi_{j-1}^\psi (p'_u(k) - p'_u(i,j-1,k))}{\delta\phi^2} \right). \end{aligned} \quad (11)$$

In writing Eq. (11), the same approximations are used as for Eq. (4) - see bullet points after Eq. (4). $\alpha_1(k)$ and $\beta(k)$ are vertical interpolation coefficients given as Eqs. (5) and (6).

Vertical preconditioning involves adding-up vertical terms from neighbouring points of Eq. (11) in the horizontal. Doing this removes the horizontal derivatives. Only the vertical terms remain,

$$
\begin{aligned}
\overline{PV}(i,j,k) = {} & \frac{f^2(1-\kappa)}{R\Pi_0\hat{\theta}_0}p'_u(k) + \\
& \frac{f^2\kappa\rho_0}{\hat{\Pi}_{0z}}\left(1+\frac{2g\Pi_{0zz}}{c_p\theta_{0z}\hat{\Pi}_{0z}^2}\right)\left(\boxed{\frac{\alpha_1(k)}{r^p(k+1)-r^p(k)}\left[\frac{\Pi_0(k+1)}{p_0(k+1)}p'_u(k+1)-\frac{\Pi_0(k)}{p_0(k)}p'_u(k)\right]}^{\,A} + \right. \\
& \left. \boxed{\frac{\beta_1(k)}{r^p(k)-r^p(k-1)}\left[\frac{\Pi_0(k)}{p_0(k)}p'_u(k)-\frac{\Pi_0(k-1)}{p_0(k-1)}p'_u(k-1)\right]}^{\,B}\right) - \\
& \frac{f^2 g\kappa\rho_0}{c_p\theta_{0z}\hat{\Pi}_{0z}^2}\frac{1}{(r^\theta(k)-r^\theta(k-1))}\left(\boxed{\frac{\Pi_0(k+1)p'_u(k+1)/p_0(k+1)-\Pi_0(k)p'_u(k)/p_0(k)}{r^p(k+1)-r^p(k)}}^{\,A} - \right. \\
& \left. \boxed{\frac{\Pi_0(k)p'_u(k)/p_0(k)-\Pi_0(k-1)p'_u(k-1)/p_0(k-1)}{r^p(k)-r^p(k-1)}}^{\,B}\right). \qquad (12)
\end{aligned}
$$

To satisfy the vertical boundary, see the text after Eq. (7).

Given $\overline{PV}$, Eq. (12) is inverted for p'_u with a tridiagonal solver for the preconditioning step.