

Approximate 'diagonal' and 'vertical-only tridiagonal' preconditioning of the PV equation

R.N. Bannister, March 2006, revised October 2007.

1. The balanced transform

For the balanced GCR solver, the following represents a simplified form of the U-transform

$$\begin{pmatrix} \xi' \\ p' \end{pmatrix} = \begin{pmatrix} \nabla^2 \\ f\rho_0 \end{pmatrix} \psi'_B + \begin{pmatrix} 0 \\ \langle p'_B \rangle \end{pmatrix}. \quad (1)$$

ψ'_B is a 3-d field where $\langle f\rho_0\psi'_B \rangle = 0$ ($\langle \cdot \rangle$ is level-by-level global mean) and $\langle p'_B \rangle$ is the level-by-level global mean balanced pressure. All quantities are stored on ψ -points. This includes PV, which has the full form

$$\begin{aligned} PV &= \frac{\theta_{0z}}{\rho_0} \xi' - \frac{f\theta_{0z}}{\rho_0^2} \rho' + \frac{f}{\rho_0} \theta'_z, \\ &= \frac{\theta_{0z}}{\rho_0} \xi' - \frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R \Pi_0 \hat{\theta}_0} p' - \frac{f\theta_{0z}}{\rho_0 \hat{\theta}_0} \overline{\frac{\theta_0}{\Pi_{0z}} \frac{\partial}{\partial z} \left(\kappa \frac{\Pi_0}{p_0} p' \right)} + \\ &\quad \frac{fg}{\rho_0 c_p \hat{\Pi}_{0z}^2} \frac{\partial^2}{\partial z^2} \left(\frac{\kappa \Pi_0}{p_0} p' \right) - \frac{2fg\Pi_{0zz}}{\rho_0 c_p \hat{\Pi}_{0z}^3} \frac{\partial}{\partial z} \left(\frac{\kappa \Pi_0}{p_0} p' \right), \end{aligned} \quad (2)$$

where the overbar and hats denote vertical interpolation from θ -levels to p -levels, and a subscript '0' indicates a reference state quantity. Inserting (1) into (2) gives

$$\begin{aligned} PV &= \frac{\theta_{0z}}{\rho_0} \nabla^2 \psi'_B - \\ &\quad \frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R \Pi_0 \hat{\theta}_0} [f\rho_0 \psi'_B + \langle p'_B \rangle] - \\ &\quad \frac{f\theta_{0z}}{\rho_0 \hat{\theta}_0} \overline{\frac{\theta_0}{\Pi_{0z}} \frac{\partial}{\partial z} \left(\kappa \frac{\Pi_0}{p_0} [f\rho_0 \psi'_B + \langle p'_B \rangle] \right)} + \\ &\quad \frac{fg}{\rho_0 c_p \hat{\Pi}_{0z}^2} \frac{\partial^2}{\partial z^2} \left(\frac{\kappa \Pi_0}{p_0} [f\rho_0 \psi'_B + \langle p'_B \rangle] \right) - \\ &\quad \frac{2fg\Pi_{0zz}}{\rho_0 c_p \hat{\Pi}_{0z}^3} \frac{\partial}{\partial z} \left(\frac{\kappa \Pi_0}{p_0} [f\rho_0 \psi'_B + \langle p'_B \rangle] \right). \end{aligned} \quad (3)$$

This is the PV given entirely in terms of the balanced control variables. For the bulk PV at level k , the discretization is as follows (note that all quantities are at horizontal position i, j unless otherwise stated)

$$\begin{aligned} PV(i, j, k) &= \frac{\theta_{0z}}{r^2 \rho_0 \cos \phi_j^\psi} \left(\frac{\psi'_B(i+1, j, k) - 2\psi'_B(k) + \psi'_B(i-1, j, k)}{\delta \lambda^2 \cos \phi_j^\psi} + \right. \\ &\quad \left. \frac{\cos \phi_{j+1}^p (\psi'_B(i, j+1, k) - \psi'_B(k)) - \cos \phi_j^p (\psi'_B(k) - \psi'_B(i, j-1, k))}{\delta \phi^2} \right) - \\ &\quad \frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R \Pi_0 \hat{\theta}_0} [f\rho_0(k) \psi'_B(k) + \langle p'_B \rangle(k)] - \end{aligned}$$

$$\begin{aligned}
& \frac{f\kappa}{\rho_0 \hat{\Pi}_{0z}} \left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p \hat{\Pi}_{0z}^2} \right) \left(\frac{\alpha_1(k)}{r^p(k+1) - r^p(k)} \left[\frac{\Pi_0(k+1)}{p_0(k+1)} [f\rho_0(k+1)\psi'_B(k+1) + \langle p'_B \rangle(k+1)] - \right. \right. \\
& \quad \left. \left. \frac{\Pi_0(k)}{p_0(k)} [f\rho_0(k)\psi'_B(k) + \langle p'_B \rangle(k)] \right] + \right. \\
& \quad \left. \frac{\beta_1(k)}{r^p(k) - r^p(k-1)} \left[\frac{\Pi_0(k)}{p_0(k)} [f\rho_0(k)\psi'_B(k) + \langle p'_B \rangle(k)] - \right. \right. \\
& \quad \left. \left. \frac{\Pi_0(k-1)}{p_0(k-1)} [f\rho_0(k-1)\psi'_B(k-1) + \langle p'_B \rangle(k-1)] \right] \right) + \\
& \quad \frac{fg\kappa}{\rho_0 c_p \hat{\Pi}_{0z}^2} \frac{1}{r^\theta(k) - r^\theta(k-1)} \times \\
& \left(\frac{\Pi_0(k+1)[f\rho_0(k+1)\psi'_B(k+1) + \langle p'_B \rangle(k+1)]/p_0(k+1) - \Pi_0(k)[f\rho_0(k)\psi'_B(k) + \langle p'_B \rangle(k)]/p_0(k)}{r^p(k+1) - r^p(k)} - \right. \\
& \quad \left. \frac{\Pi_0(k)[f\rho_0(k)\psi'_B(k) + \langle p'_B \rangle(k)]/p_0(k) - \Pi_0(k-1)[f\rho_0(k-1)\psi'_B(k-1) + \langle p'_B \rangle(k-1)]/p_0(k-1)}{r^p(k) - r^p(k-1)} \right)
\end{aligned} \tag{4}$$

In writing Eq. (4), for simplicity:

- Horizontal interpolation of reference state quantities to ψ -points is ignored for preconditioning. Each value is taken at its 'home' point that has the same horizontal index as the ψ -point.
- Some quantities that are part of a compound vertical interpolation can be approximately 'removed' from the compound and cancel with individual terms outside. For example (note that both overbar and hat indicate vertical interpolation)

$$\frac{f^2 \theta_{0z}}{\rho_0 \hat{\theta}_0} \frac{\theta_0}{\Pi_{0z}} \frac{\partial}{\partial z} \left(\kappa \frac{\Pi_0 \rho_0}{p_0} \psi'_B \right) \approx \frac{f^2 \theta_{0z}}{\rho_0} \frac{1}{\Pi_{0z}} \frac{\partial}{\partial z} \left(\kappa \frac{\Pi_0 \rho_0}{p_0} \psi'_B \right).$$

$\alpha_1(k)$ and $\beta_1(k)$ are vertical interpolation coefficients,

$$\alpha_1(k) = \frac{r^p(k) - r^\theta(k-1)}{r^\theta(k) - r^\theta(k-1)}, \tag{5}$$

$$\beta_1(k) = \frac{r^\theta(k) - r^p(k)}{r^\theta(k) - r^\theta(k-1)}. \tag{6}$$

1.1 Diagonal preconditioning of the balanced equation

Diagonal preconditioning involves ignoring terms in the right-hand-side of (4) that are different from position (i, j, k) . This gives

$$\begin{aligned}
PV(i, j, k) \approx & -\frac{\theta_{0z}}{r^2 \rho_0 \cos \phi_j^\psi} \left(\frac{2}{\delta \lambda^2 \cos \phi_j^\psi} + \frac{\cos \phi_{j+1}^p + \cos \phi_j^p}{\delta \phi^2} \right) \psi'_B(k) - \\
& \frac{f \theta_{0z} (1 - \kappa)}{\rho_0^2 R \Pi_0 \hat{\theta}_0} [f\rho_0(k)\psi'_B(k) + \langle p'_B \rangle(k)] - \\
& \frac{f\kappa}{\rho_0 \hat{\Pi}_{0z}} \left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p \hat{\Pi}_{0z}^2} \right) \left(\frac{A\alpha_1(k)}{r^p(k+1) - r^p(k)} + \frac{B\beta_1(k)}{r^p(k) - r^p(k-1)} \right) \frac{\Pi_0(k)}{p_0(k)} [f\rho_0(k)\psi'_B(k) + \langle p'_B \rangle(k)] -
\end{aligned}$$

$$\frac{fg\kappa}{\rho_0 c_p \hat{\Pi}_{0z}^2} \frac{1}{r^\theta(k) - r^\theta(k-1)} \times \left(\frac{A}{r^p(k+1) - r^p(k)} + \frac{B}{r^p(k) - r^p(k-1)} \right) \frac{\Pi_0(k)}{p_0(k)} [f\rho_0(k)\psi'_B(k) + \langle p'_B \rangle(k)]. \quad (7)$$

The extra factors A and B that appear in (7) are unity except in the following circumstances:

- at the top of the domain ($k = N$), $A = 0$ and
- at the bottom of the domain ($k = 1$), $B = 0$.

This is an application of the Neumann boundary conditions. Equation (7) can be parametrised in the following way (at each horizontal position)

$$PV(k) \approx (\lambda + \mu f \rho_0) \psi'_B(k) + \mu \langle p'_B \rangle(k), \quad (8)$$

where, from (7)

$$\lambda = -\frac{\theta_{0z}}{r^2 \rho_0 \cos \phi_j^\psi} \left(\frac{2}{\delta \lambda^2 \cos \phi_j^\psi} + \frac{\cos \phi_{j+1}^p + \cos \phi_j^p}{\delta \phi^2} \right), \quad (9)$$

$$\mu = -\frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R \Pi_0 \hat{\theta}_0} - \frac{f\kappa}{\rho_0 \hat{\Pi}_{0z}} \left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p \hat{\Pi}_{0z}^2} \right) \left(\frac{A\alpha_1(k)}{r^p(k+1) - r^p(k)} + \frac{B\beta_1(k)}{r^p(k) - r^p(k-1)} \right) \frac{\Pi_0(k)}{p_0(k)} - \frac{fg\kappa}{\rho_0 c_p \hat{\Pi}_{0z}^2} \frac{1}{r^\theta(k) - r^\theta(k-1)} \left(\frac{A}{r^p(k+1) - r^p(k)} + \frac{B}{r^p(k) - r^p(k-1)} \right) \frac{\Pi_0(k)}{p_0(k)}. \quad (10)$$

The unknowns $\psi'_B(k)$ and $\langle p'_B \rangle(k)$ are determined (noting the $\langle f\rho_0 \psi'_B(k) \rangle = 0$ requirement) by multiplying (8) by $f\rho_0 / (\lambda + \mu f \rho_0)$ and then taking the global mean

$$\left\langle \frac{f\rho_0}{\lambda + \mu f \rho_0} PV(k) \right\rangle = \langle f\rho_0 \psi'_B(k) \rangle + \left\langle \frac{f\rho_0 \mu}{\lambda + \mu f \rho_0} \right\rangle \langle p'_B \rangle(k),$$

$$\langle p'_B \rangle(k) = \left(\left\langle \frac{f\rho_0 \mu}{\lambda + \mu f \rho_0} \right\rangle \right)^{-1} \left(\left\langle \frac{f\rho_0}{\lambda + \mu f \rho_0} PV(k) \right\rangle - \langle f\rho_0 \psi'_B(k) \rangle \right), \quad (11)$$

$$\psi'_B(k) = \frac{PV(k) - \mu \langle p'_B \rangle(k)}{\lambda + \mu f \rho_0}. \quad (12)$$

1.2 Vertical preconditioning of the balanced equation

Vertical preconditioning involves adding-up vertical terms from neighbouring points of (4) in the horizontal. Doing this removes the horizontal derivatives. Only the vertical terms remain

$$PV(k) \approx \mu(k)\omega(k) + \mu^+(k)\omega(k+1) + \mu^-(k)\omega(k-1), \quad (13)$$

where

$$\mu(k) = -\frac{f\theta_{0z}(1-\kappa)}{\rho_0^2 R \Pi_0 \hat{\theta}_0} + \frac{f\kappa}{\rho_0 \hat{\Pi}_{0z}} \left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p \hat{\Pi}_{0z}^2} \right) \left(\frac{A\alpha_1(k)}{r^p(k+1) - r^p(k)} - \frac{B\beta_1(k)}{r^p(k) - r^p(k-1)} \right) \frac{\Pi_0(k)}{p_0(k)} - \frac{fg\kappa}{\rho_0 c_p \hat{\Pi}_{0z}^2} \frac{1}{r^\theta(k) - r^\theta(k-1)} \frac{\Pi_0(k)}{p_0(k)} \left(\frac{A}{r^p(k+1) - r^p(k)} + \frac{B}{r^p(k) - r^p(k-1)} \right), \quad (14)$$

$$\mu^+(k) = -\frac{f\kappa}{\rho_0 \hat{\Pi}_{0z}} \left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p \hat{\Pi}_{0z}^2} \right) \frac{A\alpha_1(k)}{r^p(k+1) - r^p(k)} \frac{\Pi_0(k+1)}{p_0(k+1)} + \frac{fg\kappa}{\rho_0 c_p \hat{\Pi}_{0z}^2} \frac{1}{r^\theta(k) - r^\theta(k-1)} \frac{\Pi_0(k+1)}{p_0(k+1)} \frac{A}{r^p(k+1) - r^p(k)}, \quad (15)$$

$$\mu^-(k) = \frac{f\kappa}{\rho_0 \hat{\Pi}_{0z}} \left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p \hat{\Pi}_{0z}^2} \right) \frac{B\beta_1(k)}{r^p(k) - r^p(k-1)} \frac{\Pi_0(k-1)}{p_0(k-1)} + \frac{fg\kappa}{\rho_0 c_p \hat{\Pi}_{0z}^2} \frac{1}{r^\theta(k) - r^\theta(k-1)} \frac{\Pi_0(k-1)}{p_0(k-1)} \frac{B}{r^p(k) - r^p(k-1)}, \quad (16)$$

$$\omega(k) = f\rho_0(k)\psi'_B(k) + \langle p'_B \rangle(k). \quad (17)$$

Equation (13) is a tridiagonal system of equations which can be solved for $\omega(k)$ at each horizontal position. Once $\omega(k)$ is known, it is then possible to derive the required fields from (17)

$$\langle p'_B \rangle(k) = \langle \omega(k) \rangle, \quad (18)$$

$$\psi'_B(k) = \frac{\omega(k) - \langle p'_B \rangle(k)}{f\rho_0(k)}. \quad (19)$$