Adjoint of 'zero adjustment' step for Poisson solver

Given ρ, we wish to find ψ by solving the Poisson equation,

$$\nabla^2\psi = \rho. \tag{1}$$

On the sphere, the right-hand side of a Poisson equation, ρ, must integrate to zero,

$$\int_{\lambda=0}^{2\pi} \mathrm{d}\lambda \int_{\phi=-\pi/2}^{\pi/2} \mathrm{d}\phi \, \cos\phi \, \rho(\lambda, \phi) = 0. \tag{2}$$

If it does not sum to zero, then it should be adjusted so that it does. Replace ρ in Eq. (1) with,

$$\rho'(\lambda, \phi) = \rho(\lambda, \phi) - C, \tag{3}$$

where C is a constant such that $\rho - C$ does sum to zero,

$$\int_{\lambda=0}^{2\pi} \mathrm{d}\lambda \int_{\phi=-\pi/2}^{\pi/2} \mathrm{d}\phi \, \cos\phi \, \big(\rho(\lambda, \phi) - C\big) = 0. \tag{4}$$

Equation (4) gives an expression for C,

$$C = \frac{\int \mathrm{d}\lambda \int \mathrm{d}\phi \, \cos\phi \, \rho(\lambda, \phi)}{\int \mathrm{d}\lambda \int \mathrm{d}\phi \, \cos\phi}, \tag{5}$$

which, in a discrete, regular longitude latitude grid approximates to,

$$C \approx \frac{\sum_i \sum_j \cos\phi_j \, \rho_{ij}}{\sum_i \sum_j \cos\phi_j}, \tag{6}$$

where the sums are over all longitude (i) and latitude (j) points on the grid. The question is, what, if anything, should be done in the adjoint step? What is the adjoint of the adjustment of Eq. (3)? By substituting Eq. (6) into Eq. (3) we can write this adjustment as,

$$\rho'_{i'j'} = \frac{1}{A} \sum_i \sum_j (A\delta_{ii'}\delta_{jj'} - \cos\phi_j) \, \rho_{ij}, \tag{7}$$

$$\text{where} \qquad A = \textstyle\sum_i \sum_j \cos\phi_j. \tag{8}$$

The chain rule relates derivatives with respect to $\rho'_{i'j'}$ with derivatives with respect to ρ_{ij},

$$\frac{\partial}{\partial\rho_{ij}} = \sum_{i'} \sum_{j'} \frac{\partial\rho'_{i'j'}}{\partial\rho_{ij}} \frac{\partial}{\partial\rho'_{i'j'}}. \tag{9}$$

The derivatives $\partial\rho'_{i'j'} / \partial\rho_{ij}$ form the matrix elements of the adjoint of the adjustment Eq. (3). Equation (7) can be used to evaluate them. Start by working out the derivative of $\rho'_{i'j'}$ with respect to an arbitrary element ρ_{mn},

$$\frac{\partial\rho'_{i'j'}}{\partial\rho_{mn}} = \frac{1}{A} \sum_i \sum_j (A\delta_{ii'}\delta_{jj'} - \cos\phi_j) \, \delta_{im}\delta_{jn},$$

$$= \frac{1}{A} (A\delta_{mi'}\delta_{nj'} - \cos\phi_n),$$

$$\therefore \qquad \frac{\partial\rho'_{i'j'}}{\partial\rho_{ij}} = \delta_{ii'}\delta_{jj'} - \frac{1}{A} \cos\phi_j. \tag{10}$$

Equation (9) can then be completed,

$$\frac{\partial}{\partial\rho_{ij}} = \frac{\partial}{\partial\rho'_{ij}} - \frac{1}{A} \sum_{i'} \sum_{j'} \cos\phi_j \frac{\partial}{\partial\rho'_{i'j'}}. \tag{11}$$

Equation (10) are matrix elements of the adjoint operator relating the derivative as in Eq. (9). We call derivatives with respect to a variable the *adjoint state* of that element and denote it with a hat. Equation (9) then becomes,

$$\hat{\rho}_{ij} \; = \; \frac{1}{A} \sum_{i'} \sum_{j'} (A\delta_{ii'}\delta_{jj'} \; - \; \cos\phi_j)\, \hat{\rho}'_{i'j'}. \tag{12}$$

Notice that Eq. (12) - the adjoint equation - resembles Eq. (7) - the forward equation - but with the primed and unprimed ρ and $\hat{\rho}$-variables swapped, and with the summation over i', j' instead of i, j. The factors connecting the $\hat{\rho}$-variables is the same as those connecting the ρ-variables.