PV of the first two vertical modes

30th March 2004 - 1st April 2004

1. The calculation

PV' definition

The definition of our PV' (as in Eq. 19 of my document and where the * subscript indicates that the perturbation ($'$) is the one used in my document, although the * is not used there) is,

$$\begin{aligned} PV'(x,y,z) &= \alpha(y,z)\zeta'_*(x,y,z) + \beta(y,z)p'_*(x,y,z) + \gamma(y,z)\Pi'_{z*}(x,y,z) + \delta(y,z)\Pi'_{zz*}(x,y,z) \\ &= -\alpha(y,z)\frac{\partial u'_*}{\partial y}(x,y,z) + \alpha(y,z)\frac{\partial v'_*}{\partial x}(x,y,z) \\ &\quad +\beta(y,z)p'_*(x,y,z) + \gamma(y,z)\Pi'_{z*}(x,y,z) + \delta(y,z)\Pi'_{zz*}(x,y,z). \end{aligned} \tag{1}$$

$\alpha,\ \beta,\ \gamma,\ \delta$ are basic state dependent prefactors (as are $\eta,\ \varepsilon,\ \mu$ below) and subscript z indicates vertical derivative. Write exner pressure derivatives in terms of ordinary pressure. The first derivative of exner is a linear combination of p'_* and p'_{z*}, and the second derivative is a linear combination of p'_*, p'_{z*} and p'_{zz*}. The last three terms of (1) then combine to give,

$$\begin{aligned} &\beta(y,z)p'_*(x,y,z) + \gamma(y,z)\Pi'_{z*}(x,y,z) + \delta(y,z)\Pi'_{zz*}(x,y,z) = \\ &\qquad \eta(y,z)p'_*(x,y,z) + \varepsilon(y,z)p'_{z*}(x,y,z) + \mu(y,z)p'_{zz*}(x,y,z). \end{aligned} \tag{2}$$

With a change of notation to that of Thuburn et al.,

$$u' = \rho_s u'_* \qquad v' = \rho_s v'_* \qquad p' = p'_*.$$

(where ρ_s is the ref. density assumed to be a fn. of z only - for simplicity), this gives for the PV' perturbation,

$$\begin{aligned} PV'(x,y,z) &= -\frac{\alpha(y,z)}{\rho_s(z)}\frac{\partial u'}{\partial y}(x,y,z) + \frac{\alpha(y,z)}{\rho_s(z)}\frac{\partial v'}{\partial x}(x,y,z) + \\ &\quad \eta(y,z)p'(x,y,z) + \varepsilon(y,z)p'_z(x,y,z) + \mu(y,z)p'_{zz}(x,y,z). \end{aligned} \tag{3}$$

Horizontal normal modes

Next substitute into (3) the following from Thuburn et al.,

$$\begin{pmatrix} PV'(x,y,z) \\ u'(x,y,z) \\ v'(x,y,z) \\ p'(x,y,z) \end{pmatrix} = \begin{pmatrix} \hat{PV}_m(y,z) \\ \hat{u}_m(y,z) \\ i\hat{v}_m(y,z) \\ \hat{p}_m(y,z) \end{pmatrix} \exp i(m\lambda - \sigma t),$$

giving, for zonal mode m (integer), and noting that $\partial/\partial x = (1/r\cos\phi)\ \partial/\partial\lambda$,

$$\begin{aligned} \hat{PV}_m(y,z) &= -\frac{\alpha(y,z)}{\rho_s(z)}\frac{\partial \hat{u}_m}{\partial y}(y,z) - \frac{\alpha(y,z)\,m}{\rho_s(z)\,r\cos\phi}\hat{v}_m(y,z) + \\ &\quad \eta(y,z)\hat{p}_m(y,z) + \varepsilon(y,z)\frac{\partial \hat{p}_m}{\partial z}(y,z) + \mu(y,z)\frac{\partial^2 \hat{p}_m}{\partial z^2}(y,z). \end{aligned} \tag{4}$$

Vertical normal modes

Next make the statement of separability between the vertical and horizontal for each normal mode. Since I am dealing with general data then I write as a linear combination of separable functions with the following assumed form,

$$\begin{pmatrix} \hat{u}_m(y,\ z) \\ \hat{v}_m(y,\ z) \\ \hat{p}_m(y,\ z) \end{pmatrix} \ = \ \sum_i \begin{pmatrix} \tilde{u}^i_m(y) \\ \tilde{v}^i_m(y) \\ \tilde{p}^i_m(y) \end{pmatrix} Z^i_1(z)\, e^{z/2H}. \tag{5}$$

This gives,

$$\begin{aligned} \hat{PV}_m(y,\ z) \ = \ & -\left\{\frac{\alpha(y,\ z)}{\rho_s(z)}\right\} \sum_i \frac{\partial \tilde{u}^i_m(y)}{\partial y} Z^i_1(z)\, e^{z/2H} \ - \\ & \left\{\frac{\alpha(y,\ z)}{\rho_s(z)}\right\} \frac{1}{r\ \cos\phi} m \sum_i \tilde{v}^i_m(y)\, Z^i_1(z)\, e^{z/2H} \ + \\ & \{\eta(y,\ z)\} \sum_i \tilde{p}^i_m(y)\, Z^i_1(z)\, e^{z/2H} \ + \\ & \{\varepsilon(y,\ z)\} \sum_i \tilde{p}^i_m(y) \frac{\partial\, (Z^i_1(z)\, e^{z/2H})}{\partial z} \ + \\ & \{\mu(y,\ z)\} \sum_i \tilde{p}^i_m(y) \frac{\partial^2\, (Z^i_1(z)\, e^{z/2H})}{\partial z^2}. \end{aligned} \tag{6}$$

I take the vertical normal modes to have the following orthogonality,

$$\int \mathrm{d}\, z Z^i_1(z)\, e^{z/2H} Z^j_1(z)\, e^{z/2H} \ = \ \delta_{ij} \tag{7}$$

Difficulties

I would have liked to have done an inner product of (6) with the external and first internal modes, but there are two aspects that prevent this:

1 The prefactors ($\alpha,\ \eta,\ \varepsilon,\ \mu$) are functions of height and so the orthogonality, (7), between the vertical normal modes is not useful for this.
2 Even if there were no height dependent prefactors, the vertical derivatives in the last two terms of the RHS may disrupt the orthogonality.

The prefactors are each comprised of many terms (absorbed into the $\alpha,\ \eta,\ \varepsilon,\ \mu$ in (6)). I have noticed that for an isothermal atmosphere, every component of every term of the factors inside the curly brackets in (6) has the following scaling with height,

$$\{\} \ \sim \ \exp((\kappa \ + \ 2)\, z / H). \tag{8}$$

Hence I can (approximately) remove the height dependence by dividing by this throughout, and hence overcome 1 above.

Given that the analytic forms of the normal modes given in Thuburn et al. are sines and cosines multiplied by $\exp(-z / 2H)$, this means that the $\partial^n\, (Z^i_1(z)\, e^{z/2H}) / \partial z^n$ will be orthogonal to $Z^i_1(z)\, e^{z/2H}$. This should overcome 2 above.

Performing the inner products

Multiplying (6) by $\Gamma(z) \ = \ \exp(-(\kappa \ + \ 2)\, z / H)$ to remove the height dependence of the prefactors, multiplying by $Z^j_1(z)\, e^{z/2H}$ and integrating over the height of the atmosphere gives,

$$\begin{aligned} \int \mathrm{d}z\, \hat{PV}_m(y,\ z)\, \Gamma(z)\, Z^j_1(z)\, e^{z/2H} \ = \ & -\left\{\frac{\alpha(y,\ z)}{\rho_s(z)}\Gamma\right\} \frac{\partial \tilde{u}^j_m(y)}{\partial y} \ - \\ & \left\{\frac{\alpha(y,\ z)}{\rho_s(z)}\Gamma\right\} \frac{1}{r\ \cos\phi} m \tilde{v}^j_m(y) \ + \\ & \{\eta(y,\ z)\,\Gamma\}\, \tilde{p}^j_m(y) \ + \end{aligned}$$

$$
\begin{aligned}
&\{\varepsilon(y,z)\,\Gamma\}\,\tilde{p}_m^j(y)\int \mathrm{d}z\,\frac{\partial\left(Z_1^j(z)\,e^{z/2H}\right)}{\partial z} Z_1^j(z)\,e^{z/2H} \; + \\
&\{\mu(y,z)\,\Gamma\}\,\tilde{p}_m^j(y)\int \mathrm{d}z\,\frac{\partial^2\left(Z_1^j(z)\,e^{z/2H}\right)}{\partial z^2} Z_1^j(z)\,e^{z/2H}. \qquad (9)
\end{aligned}
$$

Where, from (5) using orthogonality (7),

$$
\begin{aligned}
\tilde{u}_m^j(y) &= \int \mathrm{d}z\,\hat{u}_m(y,z)Z_1^j(z)\,e^{z/2H} \\
\tilde{v}_m^j(y) &= \int \mathrm{d}z\,\hat{v}_m(y,z)Z_1^j(z)\,e^{z/2H} \\
\tilde{p}_m^j(y) &= \int \mathrm{d}z\,\hat{p}_m(y,z)Z_1^j(z)\,e^{z/2H}
\end{aligned}
$$

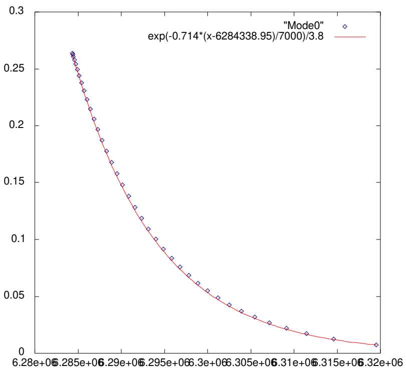
For the calculations that we wish to do, the key is the left hand side of (9). The above analysis suggests that the left hand side of Eq. (9) is the PV' associated with the jth normal mode.

2. Are the normal modes orthogonal?

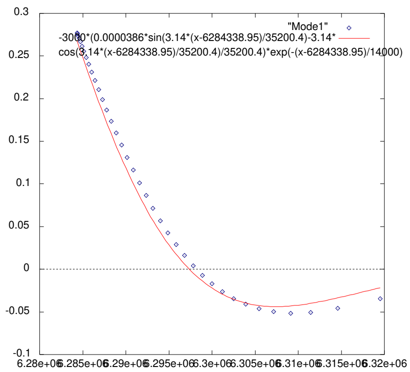
Testing the orthogonality property (7), I get the following for the first four modes,

	External	Internal 1	Internal 2	Internal 3
External	1.0	0.12	0.012	0.0012
Internal 1	0.12	1.0	0.013	0.0013
Internal 2	0.012	0.013	1.0	0.14
Internal 3	0.0012	0.0013	0.14	1.0

To check that the modes that I have calculated agree with the analytic forms in Thuburn et al., here are the numeric and analytic forms of the external and first internal modes compared (height is along the x-axis, the points are from the numerical calculation and the line is the analytic solution),



External mode



First internal mode