# The Legendre Transform

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## Orthogonality of the Legendre polynomials

The Legendre polynomials satisfy the following orthogonality property [1],

$$\int_{x=-1}^{1} \mathrm{d}x \, P_n(x) P_m(x) = \frac{2}{2n+1}\delta_{mn}, \tag{1}$$

where $P_n(x)$ is the $n$th order Legendre polynomial. In meteorology it is sometimes convenient to integrate over the latitude domain, $\phi$, instead of over $x$. This is achieved with the following transform,

$$x = \sin\phi, \qquad \frac{\mathrm{d}x}{\mathrm{d}\phi} = \cos\phi, \qquad \therefore \, \mathrm{d}x = \cos\phi \, \mathrm{d}\phi. \tag{2}$$

In latitude (measured in radians), Eq. (1) becomes,

$$\int_{\phi=-\pi/2}^{\pi/2} \mathrm{d}\phi \, \cos\phi \, P_n(\phi) P_m(\phi) = \frac{2}{2n+1}\delta_{mn}. \tag{3}$$

The Legendre transform and its inverse convert to and from the latitudinal and Legendre polynomial representations of functions (I call this these transforms a Legendre transform pair). In the following we shall develop these transforms in matrix form. The matrix form of the transforms is a compact and convenient means of performing summations over $\phi$ and over $n$ (we shall assume that functions are represented discreetly in $\phi$ space and so integrals are represented by summations).

In the notation to follow, functions are represented as vectors. A function in $\phi$ space shall be denoted by the vector $\mathsf{x}$, and a function in $n$ space shall be denoted by $\mathsf{p}$. Let the matrix consisting of columns of Legendre polynomials be $\mathbf{F}$. The orthogonality condition, Eq. (3), in matrix form is,

$$\mathbf{F}^T\mathbf{P}\mathbf{F} = \Lambda, \tag{4}$$

where $\mathbf{P}$ is the so-called diagonal inner product matrix whose diagonal elements are $\cos\phi$ and $\Lambda$ is the diagonal matrix whose diagonal elements are the normalizations $2/(2n+1)$.

## Legendre transform pair ('unbalanced' version)

The orthogonality condition, Eq. (4), allows inverse relations to be constructed easily. One choice of transform pair, in matrix form is the following,

$$\mathsf{x} = \mathbf{F}\mathsf{p}, \tag{5}$$

$$\mathsf{p} = \Lambda^{-1}\mathbf{F}^T\mathbf{P}\mathsf{x}. \tag{6}$$

Transform (6) can be confirmed by substituting Eq. (5) into Eq. (6) and confirming that the string of operators that results is, by Eq. (4), the identity matrix.

## Legendre transform pair ('balanced' version)

The transform pair, Eqs. (5) and (6) is not 'balanced'. The inverse transform, Eq. (5) involves only a projection, and the forward transform involves the inner product and the normalization matrices. This transform pair is not unique. A more 'balanced' transform pair is the following,

$$\mathsf{x} = \mathbf{P}^{1/2}\mathbf{F}\Lambda^{-1/2}\mathsf{p}, \tag{7}$$

$$\mathsf{p} = \Lambda^{-1/2}\mathbf{F}^T\mathbf{P}^{1/2}\mathsf{x}, \tag{8}$$

which can be verified using the orthogonality condition, Eq. (4).

## Gaussian quadrature with Legendre polynomials

For very good accuracy in the evaluation of integrals, the Legendre polynomials are often used to produce a Gaussian quadrature formula. We wish to evaluate an approximation to the integral,

$$A = \int_{x=-1}^{1} f(x)\, \mathrm{d}x, \tag{9}$$

when $f(x)$ is represented on a discreet set of points. In the simplest case, the points can be chosen to be $N$ equidistant abscissas. A better scheme would be to choose the abscissas $x_i$ $(1 \leqslant i \leqslant N)$ to lie on the roots of an $N$th order polynomial. The Legendre polynomial $P_N$ is a convenient choice for this in Meteorology. $P_N$ has $N$ roots, positioned at $x_i$ and correspond to special so-called Gaussian latitudes, $\phi_i$, $x_i = \sin\phi_i$. The integral is then approximated well by the Gaussian quadrature formula [2],

$$A \approx \sum_{i=1}^{N} f(x_i)\, w_i, \tag{10}$$

where the $w_i$ are the Gauss-Legendre weights. These can be found from a number of methods. [1] gives a general formula for the $w_i$ that involves integration of the polynomial $P_N$ after division by each root at $x_i$. Another approach (mentioned in [3]) exploits the orthogonality of the Legendre polynomials to construct and solve a set of linear simultaneous equations. Equation (1), written in the Gaussian quadrature is,

$$\int_{x=-1}^{1} \mathrm{d}x\, P_n(x)\, P_m(x) \approx \sum_{i=1}^{N} P_n(x_i)\, P_m(x_i)\, w_i \approx \frac{2}{2n+1}\delta_{mn}. \tag{11}$$

The set of equations for $w_i$ is constructed by setting $m = 0$ in Eq. (11). The Legendre polynomial $P_0 = 1$ and Eq. (11) becomes,

$$\sum_{i=1}^{N} P_n(x_i)\, w_i = 2\delta_{mn}, \tag{12}$$

which gives rise to the following matrix equation, which can be solved for the weights,

$$\begin{pmatrix} P_0(x_1) & P_0(x_2) & \dots & P_0(x_N) \\ P_1(x_1) & P_1(x_2) & \dots & P_1(x_N) \\ \dots & \dots & \dots & \dots \\ P_{N-1}(x_1) & P_{N-1}(x_2) & \dots & P_{N-1}(x_N) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_N \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{13}$$

[3] refers to an explicit formula that can be used instead to compute the weights for Gauss-Legendre quadrature,

$$w_i = \frac{2}{(1 - x_i^2)\,(P_N'(x_i))}. \tag{14}$$

Given the abscissas and weights, how will the Legendre transform matrix equations change? Firstly, the orthogonality Eq. (4) will be modified by the use of the Gaussian weights in the diagonal matrix **P**. Secondly, the matrix **F** will contain the Legendre polynomials (orders 0 to $N - 1$) as the respected columns, computed at the $x_i$. The matrix $\Lambda$ is unchanged and the 'balanced' transform Eqs. (7) and (8) remain valid.

## References


[1] Eric W. Weisstein, "Legendre Polynomial". From MathWorld -- A Wolfram Web Resource, http://mathworld.wolfram.com/LegendrePolynomial.html

[2] Lancoz C., Applied Analysis, Dover Publications.

[3] Press W.H., Teukolsky S.A., Vetterling W.T., Flannery B.P., Numerical Recipes, Cambridge University Press.