

Smith Cloud Scheme

Variables

T_L liquid water temperature

q_W total water (all phases)

q_C grid box mean cloud condensate. This quantity must be non-negative.

Q_C mode of cloud distribution. This quantity can be positive, zero or negative.

s local deviation from Q_C

q'_C local cloud water content (note - although this variable is primed, it is not a perturbation)

C cloud fraction

Cloud conserved variables

$$T_L = T - Lq_C / c_p \quad (1)$$

$$q_W = q + q_C \quad (2)$$

Prognostic equations

$$\frac{dq_C}{dt} = (\dot{q}_C)_A + (\dot{q}_C)_D + (\dot{q}_C)_{TM} + (\dot{q}_C)_S + (\dot{q}_C)_P + (\dot{q}_C)_{CV} \quad (3)$$

$$\frac{dq}{dt} = (\dot{q})_A + (\dot{q})_D + (\dot{q})_{TM} + (\dot{q})_S + (\dot{q})_P + (\dot{q})_{CV} \quad (4)$$

$$\frac{dT}{dt} = (\dot{T})_A + (\dot{T})_D + (\dot{T})_M + (\dot{T})_R + (\dot{T})_{TM} + (\dot{T})_S + (\dot{T})_E + (\dot{T})_{MF} + (\dot{T})_{CV} \quad (5)$$

Notes for cloud conserved variables

$$(\dot{q}_W)_S = (\dot{q}_C)_S + (\dot{q})_S \equiv 0 \quad (6)$$

$$(\dot{T}_L)_S = (\dot{T})_S - L(\dot{q}_C)_S \equiv 0 \quad (7)$$

Equation (2.19) - the grid box mean cloud water

$$Q_C = q_W - q_{SAT}(T) \quad (8)$$

Assume that T is not know, only q_W and T_W are known. From (1) and (8)

$$Q_C = q_W - q_{SAT}(T_L + LQ_C / c_p) \quad (9)$$

Solve a linearised version of this equation for Q_C

$$Q_C \approx q_W - q_{SAT}(T_L) - \left. \frac{dq_{SAT}}{dT} \right|_{T_L} \frac{LQ_C}{c_p}$$

$$\therefore Q_C \approx a_L (q_W - q_{SAT}(T_L)) \quad (10)$$

$$a_L = \left(1 + \left. \frac{dq_{SAT}}{dT} \right|_{T_L} \frac{L}{c_p} \right)^{-1} \quad (11)$$

Equation (2.20) - the perturbation to the grid box mean cloud water

Perturb the known variables (q_W and T_W) by q'_W and T'_W . What is the perturbation in Q_C , s ?

$$\begin{aligned} s = Q'_C &= a_L (q_W + q'_W - q_{SAT}(T_L + T'_L)) - a_L (q_W - q_{SAT}(T_L)) \\ &= a_L q'_W - a_L q_{SAT}(T_L + T'_L) + a_L q_{SAT}(T_L) \end{aligned}$$

$$\begin{aligned}
&\approx a_L q'_W - a_L q_{SAT}(T_L) - a_L \left. \frac{d q_{SAT}}{d T} \right|_{T_L} T'_L + a_L q_{SAT}(T_L) \\
&\approx a_L q'_W - a_L \left. \frac{d q_{SAT}}{d T} \right|_{T_L} T'_L \\
&\approx a_L (q'_W - \alpha_L T'_L)
\end{aligned} \tag{12}$$

$$\alpha_L \equiv \left. \frac{d q_{SAT}}{d T} \right|_{T_L} \tag{13}$$

Equation (2.21) - the standard deviation of s

$$\begin{aligned}
\sigma_s &= \left(\overline{(s - \bar{s})^2} \right)^{1/2} = \left(\overline{(a_L (q'_W - \alpha_L T'_L))^2} \right)^{1/2} \\
&= a_L \left(\overline{q'^2_W} + \alpha_L^2 \overline{T'^2_L} - 2\alpha_L \overline{q'_W T'_L} \right)^{1/2}
\end{aligned} \tag{14}$$

Equation (2.22) - local cloud water content

$$q'_C = Q_C + s \tag{15}$$

as long as q'_C is non-negative. Set to zero if negative - this leads to Eq. (2.22).

Equation (2.23) - mean cloud fraction

This is the fraction of the volume of the grid box that has cloud. This could be defined via the volume integral

$$C = \frac{\int dV \delta(q'_C > 0)}{\int dV} \quad \delta(q'_C > 0) \equiv \begin{cases} 0 & \text{for } q'_C = 0 \\ 1 & \text{for } q'_C > 0 \end{cases} \tag{16}$$

Let the normalised distribution function for cloud be $G(s)$. Then

$$C = \int_{-\infty}^{\infty} ds G(s) \delta(q'_C > 0)$$

where the δ -function is present to allow only those s that have non-zero q'_C . This can be achieved by adjusting the integration limits

$$C = \int_{-Q_C}^{\infty} ds G(s) \tag{17}$$

Equation (2.24) - the grid box mean cloud water content

$$q_C = \int_{-Q_C}^{\infty} ds q'_C G(s) = \int_{-Q_C}^{\infty} ds (Q_C + s) G(s) \tag{18}$$

N.B. What is the difference between Q_C and q_C ?

What is the standard deviation of G?

The functional form of G

$$G(s) = \begin{cases} s / (6\sigma_s^2) + 1 / (\sqrt{6}\sigma_s) & -\sqrt{6}\sigma_s \leq s \leq 0 \\ -s / (6\sigma_s^2) + 1 / (\sqrt{6}\sigma_s) & 0 < s \leq \sqrt{6}\sigma_s \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

$$\begin{aligned}
\sigma_s &= \left((s - \bar{s})^2 \right)^{1/2} = \left(\int_{s=-\infty}^{\infty} ds s^2 G(s) \right)^{1/2} \\
&= \left(\int_{s=-\sqrt{6}\sigma_s}^0 ds s^2 \left(\frac{s}{6\sigma_s^2} + \frac{1}{\sqrt{6}\sigma_s} \right) + \int_{s=0}^{\sqrt{6}\sigma_s} ds s^2 \left(-\frac{s}{6\sigma_s^2} + \frac{1}{\sqrt{6}\sigma_s} \right) \right)^{1/2} \\
&= \left(\left[\frac{s^4}{24\sigma_s^2} + \frac{s^3}{3\sqrt{6}\sigma_s} \right]_{-\sqrt{6}\sigma_s}^0 + \left[-\frac{s^4}{24\sigma_s^2} + \frac{s^3}{3\sqrt{6}\sigma_s} \right]_0^{\sqrt{6}\sigma_s} \right)^{1/2} \\
&= \left(-\left(\frac{36\sigma_s^4}{24\sigma_s^2} - \frac{6\sqrt{6}\sigma_s^3}{3\sqrt{6}\sigma_s} \right) + \left(-\frac{36\sigma_s^4}{24\sigma_s^2} + \frac{6\sqrt{6}\sigma_s^3}{3\sqrt{6}\sigma_s} \right) \right)^{1/2} \\
&= (4\sigma_s^2 - 3\sigma_s^2)^{1/2} = \sigma_s
\end{aligned} \tag{20}$$

This confirms the form of $G(s)$ in Fig. 1 of the paper.

Equation (2.25) - the critical relative humidity

Cloud just forms when q'_C is just zero everywhere. Insert $q'_C = 0$ and $s = \sqrt{6}\sigma_s$ into (15). This gives $Q_C = -\sqrt{6}\sigma_s$. Since there is no condensate, all water is vapour, $q = q_w$. The relative humidity under these conditions is RH_C . From (10)

$$\begin{aligned}
-\sqrt{6}\sigma_s &= a_L (q - q_{SAT}(T_L)) \\
\frac{\sqrt{6}\sigma_s}{q_{SAT}(T_L)} &= a_L \left(1 - \frac{q}{q_{SAT}(T_L)} \right) \\
&= a_L (1 - RH_C) \\
RH_C &= 1 - \frac{\sqrt{6}\sigma_s}{a_L q_{SAT}(T_L)}
\end{aligned} \tag{21}$$

Equations (2.26) to (2.28) - to make the critical relative humidity independent of temperature

As it stands, (21) is dependent upon temperature (through q_{SAT} and a_L). We may choose σ_s such that this is not the case. Choose

$$\sigma_s = A a_L q_{SAT}(T_L) f(q_w / q_{SAT}(T_L)) \tag{22}$$

where $q_w / q_{SAT}(T_L) = RH_C$ at this point, A is a constant and f is an arbitrary function. A follows from (21) and (22)

$$\begin{aligned}
RH_C &= 1 - \sqrt{6} A f(q_w / q_{SAT}(T_L)) \\
A &= \frac{1 - RH_C}{\sqrt{6} f(q_w / q_{SAT}(T_L))}
\end{aligned} \tag{23}$$

For simplicity, the paper chooses $f = 1$ leading (22) to

$$\sigma_s = \frac{a_L q_{SAT}(T_L) (1 - RH_C)}{\sqrt{6}} \tag{24}$$