Measures of observation impact in non-Gaussian data assimilation
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Abstract
Here we study the effect of relaxing the assumption of Gaussian errors on the impact of the observations on the analysis.

Introduction
It has traditionally been assumed in data assimilation (DA) that the error distributions of the prior and likelihood are Gaussian. However, there are many different potential sources of non-Gaussianity, e.g., if the prior information has come from a highly non-linear forecast model. Approximating the error distributions as Gaussian can have a detrimental impact on the accuracy of the analysis. Thus, there has been much interest in developing non-Gaussian propagation methods for the Gausses. Many of these methods are based on the direct application of Bayes’ theorem to find the full posterior distribution, $p(xy)$:

$$p(xy) = \frac{p(x|y)p(y)}{\int p(x|y)p(y)dx}$$

In applying Bayes’ theorem there is no restriction on the choice of prior, $p(x)$, or likelihood, $p(y|x)$. A measure of the impact of the observations is having in these new methods of DA can help to make the best use of the available data. This information can also allow us to understand the effect of approximating the error distributions as Gaussian.

Measures of observation impact
There are many measures of observation impact, here we introduce 3:

The analysis sensitivity to observations
This is given by

$$S = \frac{\partial p(x)}{\partial p(y)}$$

where $p(x)$ is the mean of the posterior distribution, our analysis, $p(y)$ is the mean of the likelihood, our observation, and $H$ is the linearized transform from observation to state space.

Relative entropy
This measures the entropy (uncertainty) of the posterior relative to the prior. It is sensitive to the change in the position and shape of the posterior.

$$RE = \int p(x|y)p(y)\ln\frac{p(x|y)p(y)}{p(x|y)p(y|x)}dx$$

Mutual information
Mutual information is a measure of the change in entropy when an observation is made. It is only sensitive to a change in the shape of the posterior.

$$MI = -\int p(x|y)p(y)\ln p(x|y)p(y)dx + \int p(y)p(x|y)p(y|x)dx$$

Note $MI = \int p(y)REdy$

Observation impact in Gaussian DA
In Gaussian data assimilation, explicit expressions can be given for each of the three measures of observation impact:

1. $S^2 = BHBP^{-1}R^{-1}$, where $P(x) = (B^{-1} + H^2R)^{-1}$ and $R$ are the analysis and observation error covariance matrices, respectively (Cardinali et al. [2004]). $B$ is the prior error covariance matrix.
2. $MI = -\frac{1}{2}\ln(Bp^{-1})$ (Rodgers [2000]).
3. $RE = -\frac{1}{2}\sum_{i=1}^{n} \ln(-\lambda_i)$, where $\lambda_i$ is the $i$th eigenvalue of $B$ and $n$ is the rank of $S$.

Sensitivity in non-Gaussian DA
The sensitivity of the analysis w.r.t to the observation can be calculated when either the likelihood is Gaussian and the prior is arbitrary or vice versa.

$$\frac{\partial p(x)}{\partial p(y)} = \int p(x|y)p(y)\frac{\partial p(y|x)}{\partial p(y)}dx$$

**Arbitrary prior, Gaussian likelihood**
In this case $BH$ is known to be $p(y|x) = H(x)^T R^{-1}$ and so it is easy to prove

$$\int p(x|y)p(y)\frac{\partial p(y|x)}{\partial p(y)}dx$$

This has the same form as in Gaussian DA but now $p(y)$ is a function of the observation value.

**Gaussian prior, Arbitrary likelihood** (but linear observation operator)
In this case $p(y)$ is known however $p(x|y)$ is not, and so we can use integration by parts to find

$$\int p(x|y)p(y)\frac{\partial p(y|x)}{\partial p(y)}dx$$

When the likelihood is Gaussian, and $P(z) = (B^{-1} + H^2R)^{-1}$, this is also the same form as in Gaussian DA.

As a function of $p(y)$, there is an interesting symmetry between $S^2(p(y))$ and $S^2(p(x))$, as one is proportional to the analysis error covariance and the other is inversely proportional to the analysis error covariance. This is illustrated in Fig. 1.

Comparison of measures in non-Gaussian DA
In Fig. 2 we compare the error in approximating the different measures of observation impact assuming Gaussian distributions as a function of $p(y)$.

Relative entropy is sensitive to both $i)$ the change in the position of the posterior compared to the prior and $ii)$ the change in its shape. The error in the change in the position when making Gaussian approximations will be zero when $p(y)$ is proportional to the error in the sensitivity away from this error. The error in the change of shape is inversely related to the error in $p(y)$. This part of the error in relative entropy is therefore inversely related to the sensitivity when the prior is non-Gaussian and directly related to the sensitivity when the likelihood is non-Gaussian. This explains the difference between Figs. 1 and 2b (non-Gaussian likelihood) and Fig. 2b (non-Gaussian likelihood).

This also means there is a difference in the error in $MI$ which is the expected value of relative entropy.

Figure 1. Sensitivity in non-Gaussian DA: A comparison between the sensitivity of the analysis to observations when the prior is non-Gaussian and when it is Gaussian. a) The likelihood and prior in case I. b) The likelihood and prior in case II. c) The sensitivities in the two cases (solid lines) along with their estimates when the prior/likelihood is approximated by a Gaussian (dashed lines).

Figure 2. A comparison between the different measures of observation impact normalized by their Gaussian approximation. a) When the prior is non-Gaussian b) When the likelihood is non-Gaussian.

Conclusions
- For a given observation value, the error in observation impact when Gaussian distributions are assumed can be very different depending on how it is measured.
- When the prior is non-Gaussian the error in relative entropy has a smaller range of values than that of the observation error in the sensitivity and vice versa when the likelihood is non-Gaussian. This can be understood by the different relationship the sensitivity has with the analysis error covariances in each of these cases (see Eqs. (1) and (2)).
- Mutual information, which is the expected relative entropy, tends to be much closer to it’s Gaussian approximation.
- However the error in the Gaussian approximation is again sensitive to the source of the non-Gaussianity.

References
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