Variational data assimilation

Practicalities and Covariance Matrices

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Reminder of the cost function and an explicit formula for the analysis

The 4DVar cost function considered before

$$J(\mathbf{x}) = \frac{1}{2}\left(\mathbf{x}_0 - \mathbf{x}_0^{\mathrm{b}}\right)^{\mathrm{T}} \mathbf{B}_0^{-1}\left(\bullet\right) + \frac{1}{2}\sum_{i=0}^{N}\left(\mathbf{y}_i - \mathscr{H}_i(\mathbf{x}_i)\right)^{\mathrm{T}} \mathbf{R}_i^{-1}\left(\bullet\right)$$

. . . or more generically

$$J(\mathbf{x}) = \frac{1}{2}\left(\mathbf{x} - \mathbf{x}^{\mathrm{b}}\right)^{\mathrm{T}} \mathbf{B}^{-1}\left(\bullet\right) + \frac{1}{2}\left(\mathbf{y} - \mathscr{H}(\mathbf{x})\right)^{\mathrm{T}} \mathbf{R}^{-1}\left(\bullet\right)$$

- The covariance matrices **B** and **R** influence the analysis profoundly.
- If $\mathscr{H}(\mathbf{x})$ is linear or weakly linear then $\mathscr{H}(\mathbf{x}^{\mathrm{b}} + \delta\mathbf{x}) \approx \mathscr{H}(\mathbf{x}^{\mathrm{b}}) + \mathbf{H}\delta\mathbf{x}$, and $\mathbf{x}^{\mathrm{a}}$ can be written explicitly:

$$\mathbf{x}^{\mathrm{a}} = \mathbf{x}^{\mathrm{b}} + \mathbf{B}\mathbf{H}^{\mathrm{T}}\left(\mathbf{R} + \mathbf{H}\mathbf{B}\mathbf{H}^{\mathrm{T}}\right)\left(\mathbf{y} - \mathscr{H}(\mathbf{x})\right)$$

Example of non-linear model

Suppose we have the following $\mathscr{M}$

$$\mathbf{x} = \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

$$\begin{aligned} \frac{d\theta}{dt} &= -\alpha(\theta - \phi)^2 \\ \frac{d\phi}{dt} &= g(\phi) + f(t) \end{aligned}$$

Linearization (tangent linear model)

$$d\theta/dt = -\alpha(\theta - \phi)^2 \qquad d\phi/dt = g(\phi) + f(t)$$
$$\theta(t) = \theta^{\mathrm{R}}(t) + \delta\theta(t) \qquad \phi(t) = \phi^{\mathrm{R}}(t) + \delta\phi(t)$$

$$d\theta^{\mathrm{R}}/dt = -\alpha(\theta^{\mathrm{R}} - \phi^{\mathrm{R}})^2 \qquad d\phi^{\mathrm{R}}/dt = g(\phi^{\mathrm{R}}) + f(t)$$
$$d\delta\theta/dt \approx -2\alpha(\theta^{\mathrm{R}} - \phi^{\mathrm{R}})(\delta\theta - \delta\phi) \qquad d\delta\phi/dt \approx \left.\frac{dg}{d\phi}\right|_{\mathrm{R}} \delta\phi + 0$$

Euler scheme: $dx/dt \Rightarrow (x(t_{i+1}) - x(t_i))/\Delta t$, let $F^{\mathrm{R}} \equiv 2\alpha(\theta^{\mathrm{R}} - \phi^{\mathrm{R}})$

$$\delta\theta(t_{i+1}) \approx \begin{array}{c}\delta\theta(t_i) - \\ \Delta t F^{\mathrm{R}}(\delta\theta(t_i) - \delta\phi(t_i))\end{array} \qquad \delta\phi(t_{i+1}) \approx \begin{array}{c}\delta\phi(t_i) + \\ \Delta t(dg/d\phi)\delta\phi(t_i)\end{array}$$

$$\begin{pmatrix} \delta\theta(t_{i+1}) \\ \delta\phi(t_{i+1}) \end{pmatrix} \approx \begin{pmatrix} 1 - \Delta t F^{\mathrm{R}} & \Delta t F^{\mathrm{R}} \\ 0 & 1 + \Delta t(dg/d\phi) \end{pmatrix} \begin{pmatrix} \delta\theta(t_i) \\ \delta\phi(t_i) \end{pmatrix}$$

Adjoint formulation

$$\delta\theta(t_{i+1}) \approx \delta\theta(t_i) - \Delta t F^{\mathrm{R}}(\delta\theta(t_i) - \delta\phi(t_i)) \qquad \delta\phi(t_{i+1}) \approx \delta\phi(t_i) + \Delta t (dg/d\phi)\delta\phi(t_i)$$

Recipe for adjoint

- $\delta\theta(t_i) := 0$
- $\delta\phi(t_i) := 0$

- $\delta\theta(t_i) := \delta\theta(t_i) + \delta\theta(t_{i+1})$
- $\delta\theta(t_i) := \delta\theta(t_i) - \Delta t F^{\mathrm{R}} \delta\theta(t_{i+1})$
- $\delta\phi(t_i) := \delta\phi(t_i) + \Delta t F^{\mathrm{R}} \delta\theta(t_{i+1})$

- $\delta\phi(t_i) := \delta\phi(t_i) + \delta\phi(t_{i+1})$
- $\delta\phi(t_i) := \delta\phi(t_i) + \Delta t (dg/d\phi)\delta\phi(t_{i+1})$

Exercise: write in matrix form and check transpose of linear model.
Remember: adjoint goes backwards!

Important tests

Bug warning

- Correctness test (TLM coded correctly?)

$$\lim_{\gamma \to 0} \frac{\left\| \mathscr{M}(\mathbf{x}^{\mathrm{R}} + \gamma \delta \mathbf{x}) - \mathscr{M}(\mathbf{x}^{\mathrm{R}}) - \mathbf{M} \gamma \delta \mathbf{x} \right\|}{\| \mathbf{M} \gamma \delta \mathbf{x} \|} \overset{?}{\to} 0$$

- Validity test (TLM a good approximation?)

$$\text{Does } \mathscr{M}(\mathbf{x}^{\mathrm{R}} + \gamma \delta \mathbf{x}) - \mathscr{M}(\mathbf{x}^{\mathrm{R}}) \text{ compare well to } \mathbf{M} \gamma \delta \mathbf{x}?$$

- Adjoint test ($\mathrm{TLM}^{\mathrm{T}}$ coded correctly?) – machine precision required!

$$(\mathbf{M}\delta\mathbf{x})^{\mathrm{T}}(\mathbf{M}\delta\mathbf{x}) \overset{?}{=} \delta\mathbf{x}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}\delta\mathbf{x}$$

- Gradient test (tests many aspects of the DA)

$$\Phi \equiv \frac{J(\mathbf{x}^{\mathrm{R}} + \gamma \hat{\mathbf{h}}) - J(\mathbf{x}^{\mathrm{R}})}{\gamma \hat{\mathbf{h}}^{\mathrm{T}} \nabla J(\mathbf{x}^{\mathrm{R}})} \overset{?}{\to} 1 + \mathscr{O}(\alpha), \qquad \hat{\mathbf{h}} = \frac{\nabla J(\mathbf{x}^{\mathrm{R}})}{\| \nabla J(\mathbf{x}^{\mathrm{R}}) \|}$$

Comments

- Care must be taken with linearizing processes that are non-differentiable (e.g. switches), and with iterative processes.
- Adjoint coding techniques applies to observation operators, $\mathbf{H}$, as well as to time evolution models, $\mathbf{M}$.
- Tangent linear and adjoint code requires a reference state, $\mathbf{x}^{\mathrm{R}}$, which needs to be stored or recalculated (can be difficult for large models).
- Automatic adjoint compilers exist (e.g. TAF, ODYSSEE, ADIFOR, Python modules).

Selected References

- Coding a TLM and adjoint

  - W.C. Chao and L-P. Chang (1992), Development of a four-dimensional variational analysis system using the adjoint method at GLA. Part I: Dynamics. Mon. Wea. Rev., 120:1661-1673.
  - R. Giering and T. Kaminski (1998), Recipes for adjoint code construction. ACM Trans. On Math. Software, 24:437-474.

- Testing a TLM and adjoint:

  - Y. Li, I.M. Navon, W. Yang, X. Zou, J.R. Bates, S. Moorthi and R.W. Higgins (1994), Four-dimensional variational data assimilation experiments with a multilevel semi-Lagrangian semi-implicit general circulation model. Mon. Wea. Rev., 122:966-983.
  - A.S. Lawless, N.K. Nichols and S.P. Ballard (2003), A comparison of two methods for developing the linearization of a shallow-water model, Quart. J. Roy. Met. Soc., 129:1237-1254.