Bayes theorem with Gaussian mixtures

Joint NCEO/ECMWF Intensive Course on Data Assimilation

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1 Objective

This activity will allow the student to visualise the application of Bayes theorem in a uni-dimensional setting. In particular, the pdf's (probability density functions) of the prior and likelihood are Gaussian mixtures.

2 Review of Theory

2.1 Bayes theorem

Let $\mathbf{x} \in \mathcal{R}^{N_x}$ be a random variable with a prior pdf $p(\mathbf{x})$. Let $\mathbf{y} \in \mathcal{R}^{N_y}$ be an observation, which is related to the state variable by the likelihood pdf $p(\mathbf{y}|\mathbf{x})$. The posterior pdf (i.e. the conditional pdf of the state variables *given* the observations) can be obtained using Bayes theorem:

$$p\left(\mathbf{x}|\mathbf{y}\right) = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{y})} = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{\int_{\mathcal{X}} p(\mathbf{y}|\mathbf{x})p(\mathbf{x})d\mathbf{x}} \tag{1}$$

where $\mathcal{X}$ is the statistical support of the variable $\mathbf{x}$. The marginal pdf $p(\mathbf{y})$ for the observations does not depend on the values of the state variables, and therefore can be considered a normalisation constant.

2.2 Gaussian mixtures

Let us work in a univariate setup. For brevity, let us denote the pdf of a Gaussian random variable x as:

$$\phi\left(x;\mu,\sigma\right) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{2}$$

where $-\infty < \mu < \infty$ is the population mean, and $\sigma > 0$ is the population standard deviation. If x has a Gaussian mixture distribution, its pdf can then be written as:

$$p(x) = \sum_{n=1}^{N} \alpha_n \phi\left(x;\mu_n,\sigma_n\right) \tag{3}$$

i.e. it is a weighted sum of individual Gaussian components. This sum needs to be convex, i.e.

$$\alpha_n \geq 0 \ \forall \ n, \quad \sum_{n=1}^{N} \alpha_n = 1 \tag{4}$$

3 Instructions

These are the python files used in this activity:

- *ControlBayes.py*. This is the control file, and it is the one which you will be running and modifying.
- *functionsBayes.py*. This file has the code to compute Gaussian mixture pdf's, as well as the pointwise operations in Bayes theorem.

You will run different sections of the file *ControlBayes.py*. These are enumerated as comments of the file (recall that in python # is used for comments). To run **only** a section of a file you can highlight the desired instructions with the mouse, and the press F9.

3.1 Instructions

- The first lines of the file import the different packages that the file uses: numpy, matplotlib, and the functions we have created for this activity.
- *Section 0*. The statistical support of the random variable is defined.
- *Section 1*. In this section you can edit the components of the prior. There is one array for the coefficients (α's), one for the means and one for the standard deviations for each component of the Gaussian mixture. Try changing the values and the number of components. Recall that the size of the three arrays has to be the same.
- *Section 2*. In this section you have to define the value of the actual observation y, as well as the components of the Gaussian mixture. This pdf is overall centred in the observation.
- *Section 3*. This small section contains the commands to compute Bayes theorem.
- *Section 4*. This section contains the instructions to plot the prior, likelihood and posterior pdf's in the same graph.