

Variational data assimilation II

Background and methods

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Consider the minimum point of the strong constraint cost function (the 'analysis')

What should we expect J_{\min} to be?

$$J(\delta x_0) = \frac{1}{2} (\delta x_0 - \delta x_0^b)^T B_0^{-1} (\bullet) + \frac{1}{2} \sum_{i=0}^T (y_i - \mathcal{H}_i(x_i^R) - H_i \delta x_i)^T R_i^{-1} (\bullet)$$

- State vector size n , number of observations p .
- Assume B , R correct, \mathcal{H} , \mathcal{M} perfect and linear, Gaussian error statistics.

1 $(n + p)/2$

2 $n/2$

3 $p/2$

4 No theoretical value

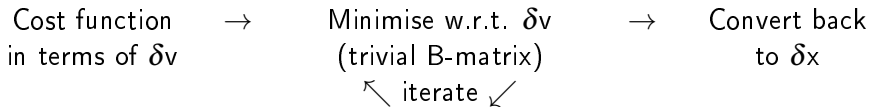
Hmmm ...

Some challenges ahead

- Methods assume that error cov. matrices are correctly known.
- Representing B_0 .
 - Better models of B_0 .
 - Flow dependency (e.g. Ensemble-Var or hybrid methods).
- Representing R_j .
 - Allowing for observation error covariances.
- Representing Q_j .
- Numerical conditioning of the problem.
- Application to more complicated systems (e.g. high-resolution models, coupled atmosphere-ocean DA, chemical DA).
- Variational bias correction.
- Moist processes, inc. clouds.
- Effective use on massively parallel computer architectures.

Making variational DA work – control variable transforms

- B_0 is an $n \times n$ matrix.
 - In operational problems B_0 is **too large** to store, let alone invert.
 - An **unknowable** matrix.
- Can **model the essential features of B_0** with a change of variable, $\delta x = U\delta v$ (a control variable transform).
 - Hypothesise that the problem is **much simpler when posed in terms of δv** rather than δx .



- Equivalent to solving original problem w.r.t. δx with $B_0 = UU^T$. See **Bannister (2008)**.

- **Illustrate in simplest case:** 3DVar with $x_0^R = x_0^b$ (and drop time index)

$$J^{3DVar}(\delta x) = \frac{1}{2} \delta x^T B^{-1} \delta x + \frac{1}{2} (y - \mathcal{H}(x^b) - H\delta x)^T R^{-1} (\bullet).$$

- **Make a change of variable:** $\delta x = U\delta v$ and assume B-matrix of errors in the δv representation is identity, I.

$$B = \langle \delta x \delta x^T \rangle_b = U \langle \delta v \delta v^T \rangle_b U^T = UIU^T = UU^T.$$

- **Substitute into $J^{3DVar}(\delta x)$:** gives a cost function w.r.t. δv

$$J^{3DVar}(\delta v) = \frac{1}{2} \delta v^T \delta v + \frac{1}{2} (y - \mathcal{H}(x^b) - HU\delta v)^T R^{-1} (\bullet)$$

$$\nabla_{\delta v} J^{3DVar} = \delta v - U^T H^T (y - \mathcal{H}(x^b) - HU\delta v)$$

$$x^a = x^b + U (\operatorname{argmin} [J^{3DVar}(\delta v)]).$$

Estimating a B-matrix

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} & \mathbf{B} &= \left\langle (\mathbf{x}^b - \mathbf{x}^t) (\mathbf{x}^b - \mathbf{x}^t)^T \right\rangle_b \\ &= \begin{pmatrix} \langle (x_1^b - x_1^t)^2 \rangle_b & \cdots & \langle (x_1^b - x_1^t)(x_n^b - x_n^t) \rangle_b \\ \vdots & \ddots & \vdots \\ \langle (x_n^b - x_n^t)(x_1^b - x_1^t) \rangle_b & \cdots & \langle (x_n^b - x_n^t)^2 \rangle_b \end{pmatrix} \end{aligned}$$

$\langle \bullet \rangle_b$: average over population of possible backgrounds.

Problem

The 'truth', \mathbf{x}^t , that appears in the definition of error, $\boldsymbol{\varepsilon} = \mathbf{x}^b - \mathbf{x}^t$, is unknowable, so need a proxy for this quantity.

Approaches to estimating a B-matrix (1)

“Canadian quick” method

$$x^b - x^t \sim (x^b(t+T) - x^b(T)) / \sqrt{2}.$$

Take population from one long time run, [Polavarapu et al. \(2005\)](#).

Approaches to estimating a B-matrix (2)

Analysis of innovations

Choose a pair of direct/independent obs locations separated by Δr :

$$\begin{aligned} & [y_r - x_r^b] [y_{r+\Delta r} - x_{r+\Delta r}^b] = \\ & [\{y_r - x_r^t\} - \{x_r^b - x_r^t\}] [\{y_{r+\Delta r} - x_{r+\Delta r}^t\} - \{x_{r+\Delta r}^b - x_{r+\Delta r}^t\}] = \\ & [\varepsilon_r^y - \varepsilon_r^b] [\varepsilon_{r+\Delta r}^y - \varepsilon_{r+\Delta r}^b]. \end{aligned}$$

Take the expectation:

$$\begin{aligned} \langle [\varepsilon_r^y - \varepsilon_r^b] [\varepsilon_{r+\Delta r}^y - \varepsilon_{r+\Delta r}^b] \rangle &= \langle \varepsilon_r^y \varepsilon_{r+\Delta r}^y \rangle + \langle \varepsilon_r^b \varepsilon_{r+\Delta r}^b \rangle \\ &= \sigma_0^2 \delta_{\Delta r, 0} + \sigma_b^2 \text{cor}_b(\Delta r). \end{aligned}$$

Above assumes obs and bg errors, as are errors between obs at different locations. Take population from many pairs with same Δr . Rutherford (1972), Hollingsworth and Lönnberg (1986), Järvinen (2001).

Approaches to estimating a B-matrix (3)

National Meteorological Center (NMC) method

Choose pairs of lagged forecasts valid at the same time, e.g.:

$$x^b - x^t \sim (x_{48}^b(t) - x_{24}^b(t)) / \sqrt{2}.$$

Take population from difference at many times. Parrish and Derber (1992), Berre et al. (2006).

Approaches to estimating a B-matrix (4)

Ensemble method

If you have an ensemble that is correctly spread:

$$x^b - x^t \sim x_{(i)}^b - \langle x^b \rangle$$

or

$$x^b - x^t \sim \left(x_{(i)}^b - x_{(j)}^b \right) / \sqrt{2}.$$

Take population from ensemble members and over many times. [Houtekamer et al. \(1996\)](#), [Buehner \(2005\)](#), [Bonavita et al. \(2015\)](#).

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