Hybrid Data Assimilation II

Hybrid methods

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influunt dependens variationis ≈ flow-dependent variational

How do we combine the properties of 'flow-dependentness' of ensemble methods with the 'full-rankness' of variational methods?

Quiz: Which of the following is a definition of a hybrid data assimilation method?

A. An ensemble DA method that uses a variational solution?

B. A method that combines the $\mathbf{B}$-matrix of Var with the $\mathbf{P}^{\mathrm{f}}$-matrix of the EnKF?

C. A method that takes the arithmetic average of the analysis increments of Var and EnKF?

D. A method that takes the geometric average of the analysis increments of Var and EnKF?

How do we combine the properties of 'flow-dependentness' of ensemble methods with the 'full-rankness' of variational methods?

Quiz: Which of the following is a definition of a hybrid data assimilation method?

A. An ensemble DA method that uses a variational solution? *This is formally called pure EnVar.*

B. A method that combines the $\mathbf{B}$-matrix of Var with the $\mathbf{P}^{\mathrm{f}}$-matrix of the EnKF? *This is formally called hybrid-EnVar.*

~~**C.** A method that takes the arithmetic average of the analysis increments of Var and EnKF?~~

~~**D.** A method that takes the geometric average of the analysis increments of Var and EnKF?~~

Pure ensemble-variational methods (EnVar), no localisation

The basic idea (initial time, no localisation)

Suppose that we have an ensemble of model states from a separate EnKF. Let an increment at $t = 0$ be a linear combination of scaled forecast ensemble perturbations at $t = 0$ (no time indices means $t = 0$).

$$\begin{aligned}
\delta\mathbf{x} &= \sum_{\ell=1}^{N} [\mathbf{v}_{\text{ens}}]_{\ell} \frac{\left(\mathbf{x}^{\text{f}(\ell)} - \overline{\mathbf{x}^{\text{f}}}\right)}{\sqrt{N-1}} \\
&= \mathbf{X}'^{\text{f}}\mathbf{v}_{\text{ens}}, \qquad \mathbf{v}_{\text{ens}} \in \mathbb{R}^{N}
\end{aligned}$$

$$\text{recall } \mathbf{X}'^{\text{f}} = \frac{1}{\sqrt{N-1}} \begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ \mathbf{x}^{\text{f}(1)} - \overline{\mathbf{x}^{\text{f}}} & \cdots & \mathbf{x}^{\text{f}(\ell)} - \overline{\mathbf{x}^{\text{f}}} & \cdots & \mathbf{x}^{\text{f}(N)} - \overline{\mathbf{x}^{\text{f}}} \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix}$$

What does this remind us of? A control variable transform: "$\delta\mathbf{x} = \mathbf{U}\mathbf{v}$", with $\mathbf{U} \to \mathbf{X}'^{\text{f}}$.

$$\text{Implied covariance } \mathbf{U}\mathbf{U}^{\text{T}} = \mathbf{X}'^{\text{f}}\mathbf{X}'^{\text{f}\,\text{T}}$$

Pure ensemble-variational methods (EnVar), no localisation

$$\delta \mathbf{x}_0 = \mathbf{X'}^{\mathrm{f}}_0 \mathbf{v}_{\mathrm{ens}}$$

En3DVar (FGAT)

Start with the incremental formulation of 3DVar-FGAT (reference state is the background)

$$J^{\mathrm{3DVar}}(\delta \mathbf{x}_0) = \frac{1}{2} \delta \mathbf{x}_0^{\mathrm{T}} \mathbf{B}^{-1} \delta \mathbf{x}_0 + \frac{1}{2} \sum_{t=-T/2}^{T/2} \left(\mathbf{y}_t - \boldsymbol{\mathcal{H}}_t(\mathbf{x}_t^{\mathrm{b}}) - \mathbf{H}_t \delta \mathbf{x}_0 \right)^{\mathrm{T}} \mathbf{R}_t^{-1} (\bullet)$$

In control variable $\mathbf{v}_{\mathrm{ens}} \in \mathbb{R}^N$ space

$$J^{\mathrm{En3DVar}}(\mathbf{v}_{\mathrm{ens}}) = \frac{1}{2} \mathbf{v}_{\mathrm{ens}}^{\mathrm{T}} \mathbf{v}_{\mathrm{ens}} + \frac{1}{2} \sum_{t=-T/2}^{T/2} \left(\mathbf{y}_t - \boldsymbol{\mathcal{H}}_t(\mathbf{x}_t^{\mathrm{b}}) - \mathbf{H}_t \mathbf{X'}^{\mathrm{f}}_0 \mathbf{v}_{\mathrm{ens}} \right)^{\mathrm{T}} \mathbf{R}_t^{-1} (\bullet)$$

$$\mathbf{x}_0^{\mathrm{a}} = \mathbf{x}_0^{\mathrm{b}} + \mathbf{X'}^{\mathrm{f}}_0 \operatorname{argmin} \left(J^{\mathrm{En3DVar}}(\mathbf{v}_{\mathrm{ens}}) \right)$$

Pure ensemble-variational methods (EnVar), no localisation

$$\delta \mathbf{x}_0 = \mathbf{X}'^{\mathrm{f}}_0 \mathbf{v}_{\mathrm{ens}}$$

En4DVar

Start with the incremental formulation of 4DVar (reference state is the background)

$$J^{\mathrm{4DVar}}(\delta \mathbf{x}_0) = \frac{1}{2} \delta \mathbf{x}_0^{\mathrm{T}} \mathbf{B}_0^{-1} \delta \mathbf{x}_0 + \frac{1}{2} \sum_{t=0}^{T} \left(\mathbf{y}_t - \boldsymbol{\mathcal{H}}_t(\mathbf{x}_t^{\mathrm{b}}) - \mathbf{H}_t \mathbf{M}_{t-1} \mathbf{M}_{t-2} \ldots \mathbf{M}_0 \delta \mathbf{x}_0 \right)^{\mathrm{T}} \mathbf{R}_t^{-1} \left(\bullet \right)$$

In control variable $\mathbf{v}_{\mathrm{ens}} \in \mathbb{R}^N$ space

$$J^{\mathrm{En4DVar}}(\mathbf{v}_{\mathrm{ens}}) = \frac{1}{2} \mathbf{v}_{\mathrm{ens}}^{\mathrm{T}} \mathbf{v}_{\mathrm{ens}} + \frac{1}{2} \sum_{t=0}^{T} \left(\mathbf{y}_t - \boldsymbol{\mathcal{H}}_t(\mathbf{x}_t^{\mathrm{b}}) - \mathbf{H}_t \mathbf{M}_{t-1} \mathbf{M}_{t-2} \ldots \mathbf{M}_0 \mathbf{X}'^{\mathrm{f}}_0 \mathbf{v}_{\mathrm{ens}} \right)^{\mathrm{T}} \mathbf{R}_t^{-1} \left(\bullet \right)$$

$$\mathbf{x}_0^{\mathrm{a}} = \mathbf{x}_0^{\mathrm{b}} + \mathbf{X}'^{\mathrm{f}}_0 \operatorname{argmin} \left(J^{\mathrm{En4DVar}}(\mathbf{v}_{\mathrm{ens}}) \right)$$

Still need the tangent linear model (and adjoint for the gradient w.r.t. $\mathbf{v}_{\mathrm{ens}}$).

Pure ensemble-variational methods (EnVar), no localisation

4DEnVar

Start with the En4DVar cost function:

$$J^{\text{En4DVar}}(\mathbf{v}_{\text{ens}}) = \frac{1}{2}\mathbf{v}_{\text{ens}}^{\text{T}}\mathbf{v}_{\text{ens}} + \frac{1}{2}\sum_{t=0}^{T}\left(\mathbf{y}_t - \boldsymbol{H}_t(\mathbf{x}_t^{\text{b}}) - \mathbf{H}_t \underbrace{\overbrace{\mathbf{M}_{t-1}\mathbf{M}_{t-2}\ldots\mathbf{M}_0}^{\mathbf{M}_{0\to t}} \mathbf{X}'^{\text{f}}_0}_{\mathbf{X}'^{\text{f}}_t} \mathbf{v}_{\text{ens}}\right)^{\text{T}} \mathbf{R}_t^{-1}(\bullet)\,.$$

Consider the ℓth column of $\mathbf{X}_0^{\text{f}}$ (call $\mathbf{x}_0^{\text{f}(\ell)}$) and do a Taylor expansion of $\boldsymbol{\mathcal{M}}_{0\to t}$:

$$\boldsymbol{\mathcal{M}}_{0\to t}\Big(\underbrace{\mathbf{x}_0^{\text{f}(\ell)}}_{\substack{\ell\text{th column}\\ \text{of } \mathbf{X}_0^{\text{f}}}}\Big) \approx \boldsymbol{\mathcal{M}}_{0\to t}\left(\overline{\mathbf{x}_t^{\text{f}}}\right) + \underbrace{\mathbf{M}_{0\to t}\Big(\underbrace{\mathbf{x}_t^{\text{f}(\ell)} - \overline{\mathbf{x}_t^{\text{f}}}}_{\ell\text{th column of } \mathbf{X}'^{\text{f}}_0}\Big)}_{\ell\text{th column of } \mathbf{X}'^{\text{f}}_t}\,.$$

$$\approx \overline{\boldsymbol{\mathcal{M}}_{0\to t}\left(\mathbf{x}_t^{\text{f}}\right)} + \mathbf{M}_{0\to t}\left(\mathbf{x}_t^{\text{f}(\ell)} - \overline{\mathbf{x}_t^{\text{f}}}\right)$$

Pure ensemble-variational methods (EnVar), no localisation

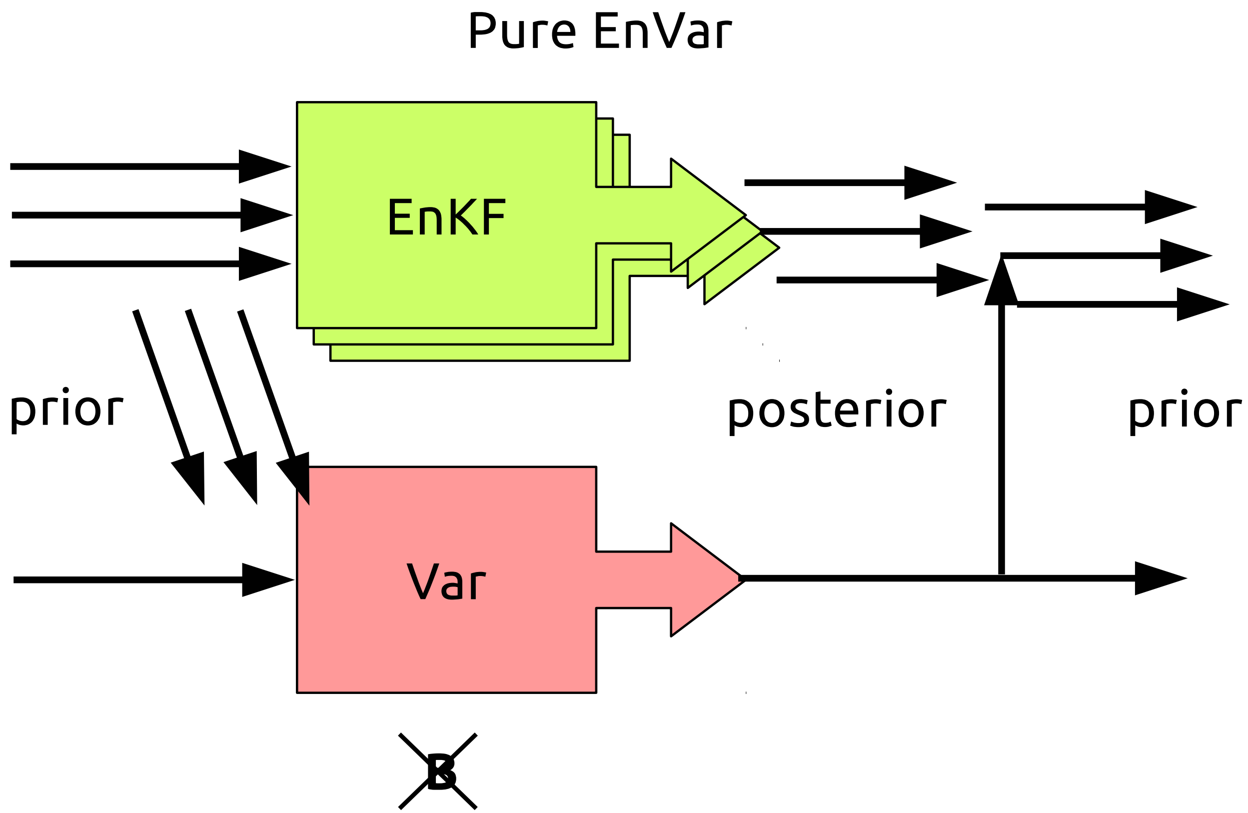
4DEnVar

$$\underbrace{\mathbf{M}_{0\to t}\left(\mathbf{x}_t^{\mathrm{f}(\ell)} - \overline{\mathbf{x}_t^{\mathrm{f}}}\right)}_{\ell\text{th column of } \mathbf{X}'^{\mathrm{f}}_t} \approx \boldsymbol{\mathcal{M}}_{0\to t}\left(\mathbf{x}_0^{\mathrm{f}(\ell)}\right) - \overline{\boldsymbol{\mathcal{M}}_{0\to t}\left(\mathbf{x}_t^{\mathrm{f}}\right)}$$

- Use the above to build columns of $\mathbf{M}_{0\to t}\mathbf{X}'^{\mathrm{f}}_0 \equiv \mathbf{X}'^{\mathrm{f}}_t$ (N runs of the model).
- This eliminates the need for a TLM (and its adjoint for the gradient).
- The method is called 4DEnVar.

$$J^{\mathrm{4DEnVar}}(\mathbf{v}_{\mathrm{ens}}) = \frac{1}{2}\mathbf{v}_{\mathrm{ens}}^{\mathrm{T}}\mathbf{v}_{\mathrm{ens}} + \frac{1}{2}\sum_{t=0}^{T}\left(\mathbf{y}_t - \boldsymbol{H}_t(\mathbf{x}_t^{\mathrm{b}}) - \mathbf{H}_t\mathbf{X}'^{\mathrm{f}}_t\mathbf{v}_{\mathrm{ens}}\right)^{\mathrm{T}}\mathbf{R}_t^{-1}\left(\bullet\right)$$

Combining En and Var ("pure EnVar")



Pure ensemble-variational methods (EnVar), localisation (loc)

- Remember localisation: $\mathbf{X}'^{\mathrm{f}}\mathbf{X}'^{\mathrm{f}\,\mathrm{T}} \to \boldsymbol{\Omega} \circ \left(\mathbf{X}'^{\mathrm{f}}\mathbf{X}'^{\mathrm{f}\,\mathrm{T}}\right)$.
- In the unlocalised system, $\mathbf{v}_{\mathrm{ens}}$ is an N-element set of scalars.
- Introduce loc: change each element of $\mathbf{v}_{\mathrm{ens}}$ from scalar $[\mathbf{v}_{\mathrm{ens}}]_\ell \to n$-element vector $\mathbf{v}_{\mathrm{ens}}^{(\ell)}$.
  - Introduce control matrix $\mathbf{V}_{\mathrm{ens}} \in \mathbb{R}^{n\times N}$ and new CVT

$$\mathbf{V}_{\mathrm{ens}} = \begin{pmatrix} \uparrow & & \uparrow \\ \mathbf{v}_{\mathrm{ens}}^{(1)} & \cdots & \mathbf{v}_{\mathrm{ens}}^{(N)} \\ \downarrow & & \downarrow \end{pmatrix}, \qquad \delta\mathbf{x} = \frac{1}{\sqrt{N-1}} \sum_{\ell=1}^{N} \left(\boldsymbol{\Omega}^{1/2}\mathbf{v}_{\mathrm{ens}}^{(\ell)}\right) \circ \left(\mathbf{x}^{\mathrm{f}(\ell)} - \overline{\mathbf{x}^{\mathrm{f}}}\right)$$

  - Let the background error covariance of each $\mathbf{v}_{\mathrm{ens}}^{(\ell)}$ be $\mathbf{I}$, and let $\mathbf{v}_{\mathrm{ens}}^{(\ell)}$ be uncorrelated with $\mathbf{v}_{\mathrm{ens}}^{(\ell')}$ $(\ell \neq \ell')$.
- Cost function with localisation (back to En3DVar e.g.)

$$J^{\mathrm{En3DVar}}(\mathbf{V}_{\mathrm{ens}}) = \frac{1}{2}\sum_{\ell=1}^{N} \mathbf{v}_{\mathrm{ens}}^{(\ell)\,\mathrm{T}} \mathbf{v}_{\mathrm{ens}}^{(\ell)} +$$

$$\frac{1}{2}\sum_{t=-T/2}^{T/2} \left(\mathbf{y}_t - \boldsymbol{\mathcal{H}}_t(\mathbf{x}_t^{\mathrm{b}}) - \mathbf{H}_t \frac{1}{\sqrt{N-1}} \sum_{\ell=1}^{N} \left(\boldsymbol{\Omega}^{1/2}\mathbf{v}_{\mathrm{ens}}^{(\ell)}\right) \circ \left(\mathbf{x}^{\mathrm{f}(\ell)} - \overline{\mathbf{x}^{\mathrm{f}}}\right)\right)^{\mathrm{T}} \mathbf{R}_t^{-1} \left(\bullet\right)$$

Pure ensemble-variational methods (EnVar), localisation

Why does this work?

$$\mathbf{V}_{\text{ens}} = \begin{pmatrix} \uparrow & & \uparrow \\ \mathbf{v}_{\text{ens}}^{(1)} & \cdots & \mathbf{v}_{\text{ens}}^{(N)} \\ \downarrow & & \downarrow \end{pmatrix} \qquad \delta\mathbf{x} = \frac{1}{\sqrt{N-1}} \sum_{\ell=1}^{N} \left(\mathbf{\Omega}^{1/2}\mathbf{v}_{\text{ens}}^{(\ell)}\right) \circ \left(\mathbf{x}^{\text{f}(\ell)} - \overline{\mathbf{x}^{\text{f}}}\right) \qquad \left\langle \mathbf{v}_{\text{ens}}^{(\ell)} \mathbf{v}_{\text{ens}}^{(\ell')\text{T}} \right\rangle_{\text{f}} = \mathbf{I}\delta_{\ell\ell'}$$

$$[\delta\mathbf{x}]_i = \frac{1}{\sqrt{N-1}} \sum_{\ell=1}^{N} (\mathbf{\Omega}^{1/2}\mathbf{v}_{\text{ens}}^{(\ell)})_i \left([\mathbf{x}^{\text{f}(\ell)}]_i - [\overline{\mathbf{x}^{\text{f}}}]_i\right)$$

$$\begin{aligned}
\langle (\delta\mathbf{x})_i \times (\delta\mathbf{x})_j \rangle_{\text{f}} &= \frac{1}{N-1} \sum_{\ell=1}^{N} \left\langle (\mathbf{\Omega}^{1/2}\mathbf{v}_{\text{ens}}^{(\ell)})_i \left([\mathbf{x}^{\text{f}(\ell)}]_i - [\overline{\mathbf{x}^{\text{f}}}]_i\right) \times \sum_{\ell'=1}^{N} (\mathbf{\Omega}^{1/2}\mathbf{v}_{\text{ens}}^{(\ell')})_j \left([\mathbf{x}^{\text{f}(\ell')}]_j - [\overline{\mathbf{x}^{\text{f}}}]_j\right) \right\rangle \\
&= \frac{1}{N-1} \sum_{\ell=1}^{N} \sum_{\ell'=1}^{N} \underbrace{\left\langle (\mathbf{\Omega}^{1/2}\mathbf{v}_{\text{ens}}^{(\ell)})_i (\mathbf{\Omega}^{1/2}\mathbf{v}_{\text{ens}}^{(\ell')})_j \right\rangle_{\text{f}}}_{\mathbf{\Omega}_{ij}\,\delta_{ll'}} \left([\mathbf{x}^{\text{f}(\ell)}]_i - [\overline{\mathbf{x}^{\text{f}}}]_i\right) \left([\mathbf{x}^{\text{f}(\ell')}]_j - [\overline{\mathbf{x}^{\text{f}}}]_j\right) \\
&= \mathbf{\Omega}_{ij} \frac{1}{N-1} \sum_{\ell=1}^{N} \left([\mathbf{x}^{\text{f}(\ell)}]_i - [\overline{\mathbf{x}^{\text{f}}}]_i\right) \left([\mathbf{x}^{\text{f}(\ell)}]_j - [\overline{\mathbf{x}^{\text{f}}}]_j\right) \\
&= \mathbf{\Omega}_{ij} \left(\mathbf{X}'^{\text{f}} \mathbf{X}'^{\text{f}\,\text{T}}\right)_{ij} \qquad \therefore \left\langle \delta\mathbf{x}\delta\mathbf{x}^{\text{T}} \right\rangle_{\text{f}} = \mathbf{\Omega} \circ \left(\mathbf{X}'^{\text{f}} \mathbf{X}'^{\text{f}\,\text{T}}\right)
\end{aligned}$$

Pure ensemble-variational methods (EnVar), localisation

Notes

- Compact form of this CVT

$$\mathbf{V} = \begin{pmatrix} \uparrow & & \uparrow \\ \mathbf{v}_{\text{ens}}^{(1)} & \cdots & \mathbf{v}_{\text{ens}}^{(N)} \\ \downarrow & & \downarrow \end{pmatrix} \qquad \delta\mathbf{x} = \frac{1}{\sqrt{N-1}} \sum_{\ell=1}^{N} \left(\boldsymbol{\Omega}^{1/2} \mathbf{v}_{\text{ens}}^{(\ell)} \right) \circ \left(\mathbf{x}^{\text{f}(\ell)} - \overline{\mathbf{x}^{\text{f}}} \right)$$

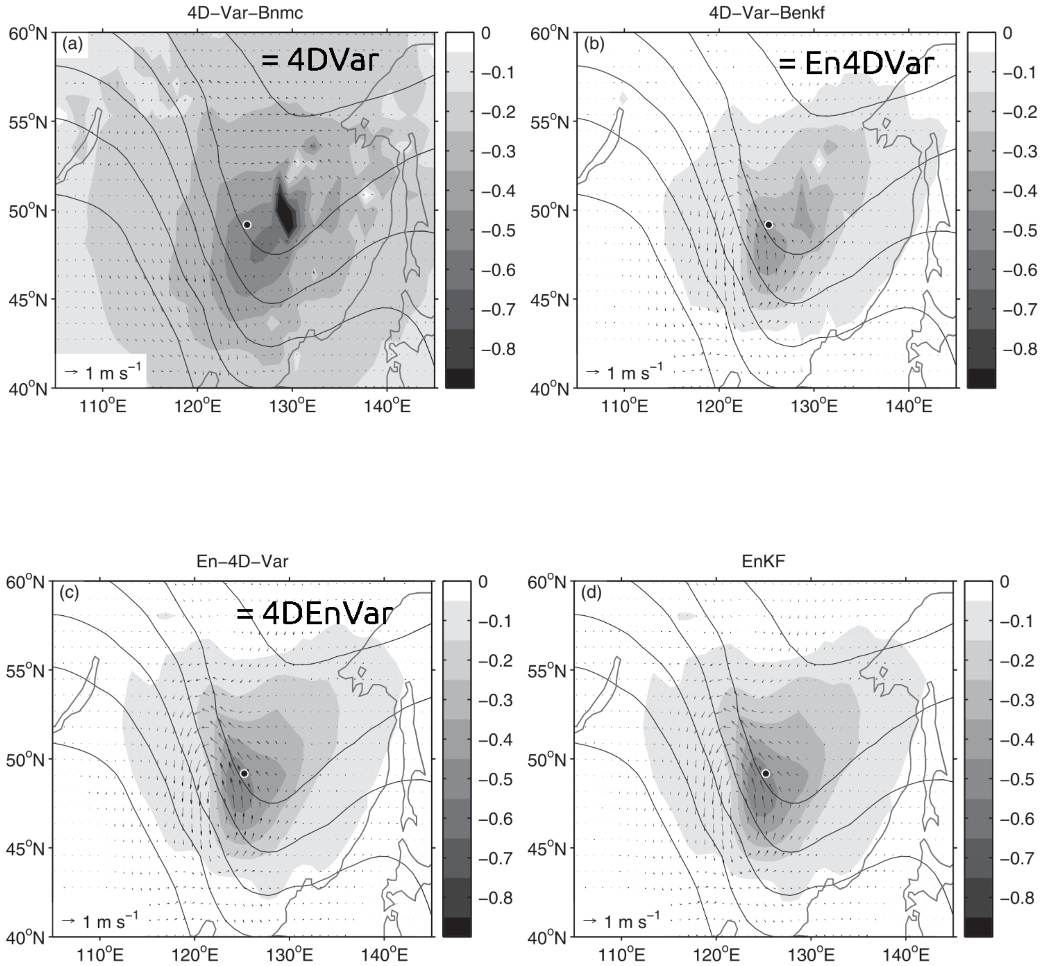
$$\delta\mathbf{x} = \left(\left(\boldsymbol{\Omega}^{1/2} \mathbf{V}_{\text{ens}} \right) \circ \mathbf{X}'^{\text{f}} \right) \mathbf{1} \qquad \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

- This is the method of localisation from Lorenc 2003 [6]. Another method is from Buehner 2005 [2].

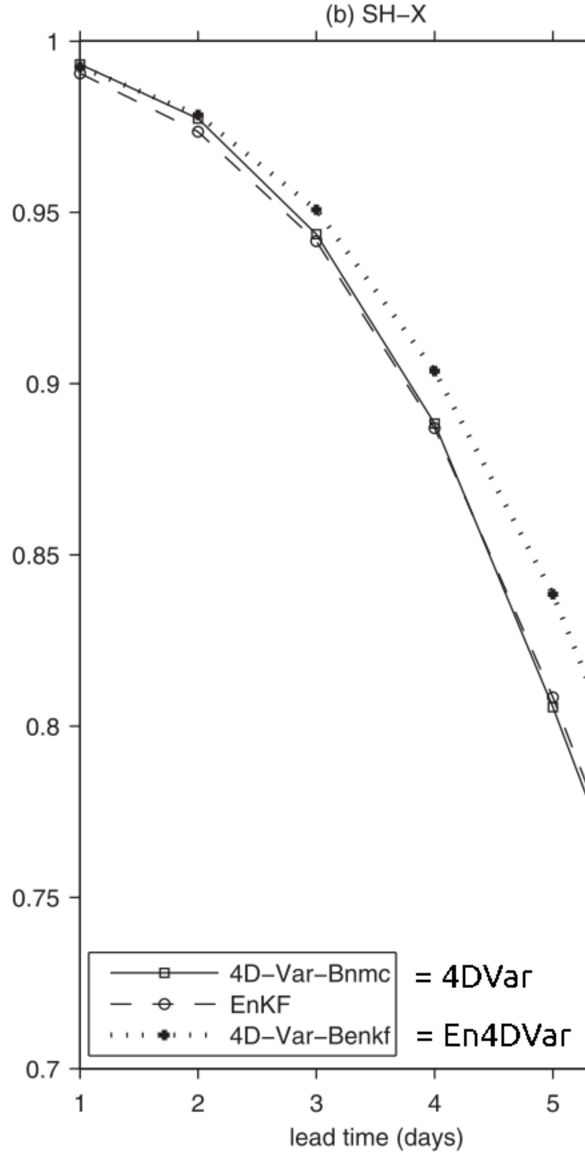
- The Schur product ($\circ$) messes-up the usual matrix manipulation.

- Don't need to represent background error covariances explicitly.

Single ob experiments and performance [3, 4]



500hPa analysis increments of T (shading) and (u, v) due to T ob $y - H(\mathbf{x}^{\mathrm{b}}) = -1\mathrm{K}$; contours are bg geopotential height.



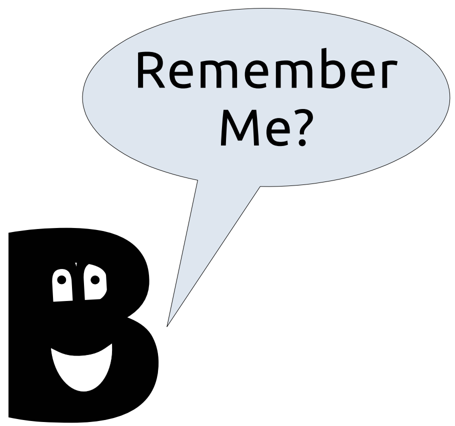
Southern hemisphere anomaly correlations for 500 hPa geopotential height.

Hybrid ensemble-variational methods (Hybrid-EnVar)

- Up to now we have only considered (localised) *ensemble-derived* covariances, $\mathbf{P}^{\mathrm{f}}$:
  - $\mathbf{P}^{\mathrm{f}}$ is flow-dependent (good), but rank deficient, etc. (bad) and not completely mitigated for with localisation.
- Remember we also have the $\mathbf{B}$-matrix from traditional variational assimilation:
  - $\mathbf{B}$ is not fully flow-dependent (bad), but can be full-rank (good), and can have some useful properties (e.g. produces nearly balanced increments).
- Propose to combine them [5]:

$$\mathbf{P}_{\mathrm{h}} = (1-\beta)\mathbf{B} + \beta\mathbf{P}^{\mathrm{f}}$$

- Trick is to represent this as a CVT.



Hybrid ensemble-variational methods (Hybrid-EnVar)

No localisation – the hybrid control variable transform

$$\delta\mathbf{x} = \underbrace{\begin{pmatrix}\sqrt{1-\beta}\,\mathbf{U} & \sqrt{\beta}\,\mathbf{X}'^{\mathrm{f}}\end{pmatrix}}_{\text{hybrid CVT}}^{\mathbf{U}_{\mathrm{h}}} \underbrace{\begin{pmatrix}\mathbf{v}_{\mathrm{B}} \\ \mathbf{v}_{\mathrm{ens}}\end{pmatrix}}_{\text{hybrid control variable}}^{\mathbf{v}_{\mathrm{h}}}$$

Implied background error covariance

$$\begin{pmatrix}\sqrt{1-\beta}\,\mathbf{U} & \sqrt{\beta}\,\mathbf{X}'^{\mathrm{f}}\end{pmatrix}\begin{pmatrix}\sqrt{1-\beta}\,\mathbf{U}^{\mathrm{T}} \\ \sqrt{\beta}\,\mathbf{X}'^{\mathrm{f}\,\mathrm{T}}\end{pmatrix} = (1-\beta)\mathbf{U}\mathbf{U}^{\mathrm{T}} + \beta\mathbf{X}'^{\mathrm{f}}\mathbf{X}'^{\mathrm{f}\,\mathrm{T}}$$

$$J^{\mathrm{hEn4DVar}}(\mathbf{v}_{\mathrm{h}}) = \frac{1}{2}\begin{pmatrix}\mathbf{v}_{\mathrm{B}}^{\mathrm{T}} & \mathbf{v}_{\mathrm{ens}}^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\mathbf{v}_{\mathrm{B}} \\ \mathbf{v}_{\mathrm{ens}}\end{pmatrix} + \frac{1}{2}\sum_{t=0}^{T}\left(\mathbf{y}_t - \boldsymbol{H}_t(\mathbf{x}_t^{\mathrm{b}}) - \mathbf{H}_t\mathbf{M}_{0\to t}\mathbf{U}_{\mathrm{h}}\mathbf{v}_{\mathrm{h}}\right)^{\mathrm{T}}\mathbf{R}_t^{-1}\left(\bullet\right)$$

Hybrid ensemble-variational methods (Hybrid-EnVar)

No localisation – the control variable transform for Hybrid En3/4DVar

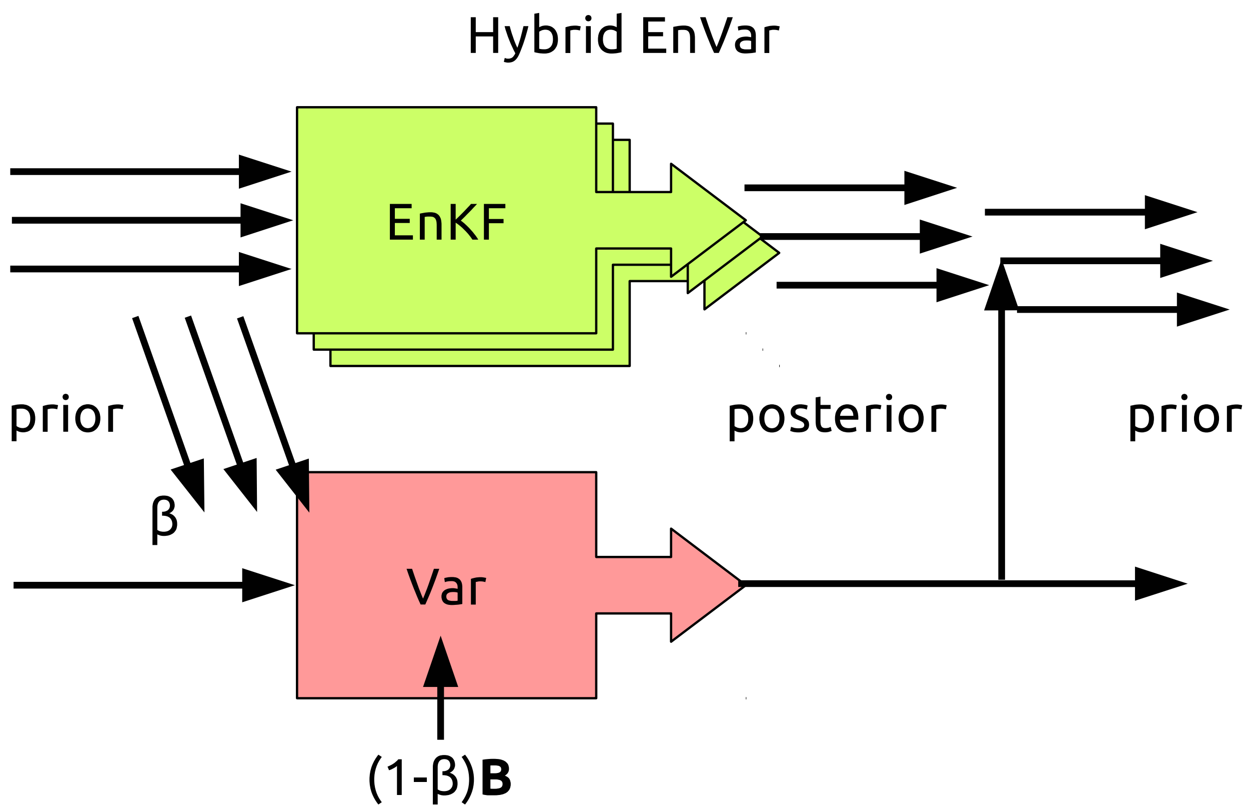
$$\delta\mathbf{x} = \sqrt{1-\beta}\,\mathbf{U}\mathbf{v}_{\mathrm{B}} + \sqrt{\beta}\,\mathbf{X}'^{\mathrm{f}}\mathbf{v}_{\mathrm{ens}} \qquad \mathbf{v}_{\mathrm{h}} = \begin{pmatrix} \mathbf{v}_{\mathrm{B}} \\ \mathbf{v}_{\mathrm{ens}} \end{pmatrix}$$

With localisation

$$\delta\mathbf{x} = \sqrt{1-\beta}\,\mathbf{U}\mathbf{v}_{\mathrm{B}} + \sqrt{\beta}\left(\mathbf{V}_{\mathrm{ens}} \circ \mathbf{X}'^{\mathrm{f}}\right)\mathbf{1} \qquad \mathbf{v}_{\mathrm{h}} = \begin{pmatrix} \mathbf{v}_{\mathrm{B}} \\ \mathbf{V}_{\mathrm{ens}} \end{pmatrix}$$

Things get complicated!

Combining En and Var with B ("hybrid EnVar")



Summary of different schemes, see e.g. [1], nomenclature [7] (* defining "Hybrid" as a mixture of static and ens-based bg err cov matrices)

All designed to improve the bg err covs (i.e. flow-dependency); all assume cvs are uncorrelated/unit variance.

Scheme	CV	CVT	Length	TLM/adj?	Comments
En3DVar	$\mathbf{v}_{\text{ens}}$	$\delta\mathbf{x}_0 = \mathbf{X}'^{\text{f}}_0\mathbf{v}_{\text{ens}}$	N	n/a	3DVar with $\mathbf{X}'^{\text{f}}\mathbf{X}'^{\text{f}\,\text{T}}$
En3DVar (loc)	$\mathbf{V}_{\text{ens}}$	$\delta\mathbf{x} = \left(\left(\boldsymbol{\Omega}^{1/2}\mathbf{V}_{\text{ens}}\right)\circ\mathbf{X}'^{\text{f}}\right)\mathbf{1}$	nN	n/a	3DVar with $\boldsymbol{\Omega}\circ\left(\mathbf{X}'^{\text{f}}\mathbf{X}'^{\text{f}\,\text{T}}\right)$
En4DVar	$\mathbf{v}_{\text{ens}}$	As En3DVar	N	Y	4DVar with $\mathbf{X}'^{\text{f}}\mathbf{X}'^{\text{f}\,\text{T}}$
En4DVar (loc)	$\mathbf{V}_{\text{ens}}$	As En3DVar (loc)	nN	Y	4DVar with $\boldsymbol{\Omega}\circ\left(\mathbf{X}'^{\text{f}}\mathbf{X}'^{\text{f}\,\text{T}}\right)$
4DEnVar	$\mathbf{v}_{\text{ens}}$	$\delta\mathbf{x}_t = \mathbf{X}'^{\text{f}}_t\mathbf{v}_{\text{ens}}$	N	N	$\mathbf{X}'^{\text{f}}_t\mathbf{X}'^{\text{f}\,\text{T}}_{t'}$
4DEnVar (loc)	$\mathbf{V}_{\text{ens}}$	4D-version of En3DVar (loc)	nN	N	4D version of En3DVar (loc)
Hybrid*-En3DVar (loc)	$\mathbf{v}_{\text{B}}, \mathbf{V}_{\text{ens}}$	$\delta\mathbf{x}_0 = \mathbf{U}\mathbf{v}_{\text{B}} + \left(\left(\boldsymbol{\Omega}^{1/2}\mathbf{V}_{\text{ens}}\right)\circ\mathbf{X}'^{\text{f}}\right)\mathbf{1}$	$n + nN$	n/a	3DVar with $(1-\beta)\mathbf{B} + \beta\boldsymbol{\Omega}\circ\mathbf{P}^{\text{f}}$
Hybrid*-En4DVar (loc)	$\mathbf{v}_{\text{B}}, \mathbf{V}_{\text{ens}}$	As Hybrid-En3DVar (loc)	$n + nN$	Y	4DVar with $(1-\beta)\mathbf{B} + \beta\boldsymbol{\Omega}\circ\mathbf{P}^{\text{f}}$
Hybrid*-4DEnVar (loc)	4D-version of Hybrid-En3DVar (loc)	4D-version of Hybrid-En3DVar (loc)	Many options	N	4D-version of Hybrid-En3DVar (loc)
Weak constraint …					

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