Dynamical systems and Data Assimilation

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Dynamical system

Consider ordinary differential equation

$$\dot{u}_t = f(t, u_t)$$

on some space $(E, (., ..))$, where

- time $t \in \mathbb{R}_{\geq 0}$,
- solution $u : \mathbb{R}_{\geq 0} \to E, t \to u_t$,
- initial value $u_0 = \xi \in E$.

We assume to have solutions for every initial value $\xi \in E$.

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Bellmann–Grönwall lemma

Suppose $\alpha : [0, T] \to \mathbb{R}_{\geq 0}$ satisfies

$$\dot{\alpha}_t \leq \lambda \alpha_t + K$$

for all $t \in [0, T]$, with $\lambda \in \mathbb{R}, K \geq 0$. Then

$$\alpha_t \leq e^{\lambda t} \alpha_0 + \frac{K}{\lambda}(e^{\lambda t} - 1)$$

Note that negative λ is permitted! In that case we have asymptotically $\alpha_t \cong \frac{K}{|\lambda|}$.

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Application 1 of Bellmann–Grönwall

Sort of energy balance

Typical GFD models look like this:

$$\dot{u}_t = \underbrace{B(u)}_{\text{Advection}} + \underbrace{Au}_{\text{Dissipation}} + \underbrace{f}_{\text{Forcing}}$$

with

- $B(u)$ Nonlinear advection, often bilinear, with the property $(u, B(u)) = 0$,
- Au Linear dissipation (viscosity) Au with the property $(u, Au) \leq -\lambda |u|^2$,
- f Forcing term, bounded $|f| \leq f_0$ (may even depend on u).

Lots of models fit that bill: Navier–Stokes, Lorenz'XX, QG, ...

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Energy balance

continued

We interprete $\frac{1}{2}|u|^2$ as *energy*. Calculate $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u|^2 = (u, \dot{u})$ and use equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u|^2 = (u, B(u)) + (u, Au) + (u, f) \le -\lambda|u|^2 + \frac{a}{2}|u|^2 + \frac{1}{2a}f_0^2.$$

(holds for arbitrary a). Take $a = \lambda$ to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u|^2 \le -\frac{\lambda}{2}|u|^2 + \frac{1}{2\lambda}f_0^2.$$

Finally, Bellmann–Grönwall gives

$$|u|^2 \le e^{-\lambda t}|u|^2 + \frac{f_0^2}{\lambda^2}(1 - e^{-\lambda t}) \to \frac{f_0^2}{\lambda^2}.$$

If solution exists at all, it remains bounded!

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More applications

Mathematical folklore theorem: Existence of solutions

Given a problem $\mathcal{P}$, suppose we can show that any would–be solution u is bounded (in a suitable sense), then $\mathcal{P}$ has a solution.

We have just done the calculations!

Other application: uniqueness of solutions

Suppose that f satisfies

$$|f(u) - f(v)| \leq \lambda |u - v| \qquad \text{for all} \qquad u, v \in E.$$

Then for two solutions u, v compute energy of $u - v$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} |u - v|^2 = (u - v, f(u) - f(v)) \leq \lambda |u - v|^2$$

so BG gives $|u - v|^2 \leq e^{2\lambda t} |u_0 - v_0|^2$.

Solutions for same initial conditions agree!

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A data assimilation problem

Suppose model has the form

$$\dot{\mathbf{Z}}_t = f(\mathbf{Z}_t) \qquad \text{with } \mathbf{Z} = (X, Y)$$

Where X hidden, Y observed. Try data assimilation with

$$\dot{\mathbf{z}}_t = f(\mathbf{z}_t) + \begin{bmatrix} 0 \\ k(Y_t - y_t) \end{bmatrix} \qquad \text{with } \mathbf{z} = (x, y).$$

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The *error* $\mathbf{e} = (e_x, e_y) = (X - x, Y - y)$ satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{e}|_t^2 = (\mathbf{e}, f(\mathbf{Z}_t) - f(\mathbf{z}_t)) - k|e_y|^2.$$

Theorem

Suppose that there are $\alpha, \beta, \gamma > 0$ so that

$$(\mathbf{e}, f(\mathbf{Z}) - f(\mathbf{z})) \leq -\alpha|e_x|^2 + \beta|e_y|^2 + \gamma|e_x||e_y|.$$

(Note the sign of α). Then $|\mathbf{e}|^2 \to 0$, provided k is set large enough

The proof is *Exercise 1*.

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Local instabilities

Consider two solutions

$$\dot{u}_t = f(u_t), \qquad \dot{v}_t = f(v_t),$$

with "nearby" initial conditions $u_0 = v_0 + h$. Can we describe $e_t := u_t - v_t$ by linearisation

$$\dot{e}_t \cong Df(u_t)e_t, \qquad e_0 = h?$$

Works well only if $(e, Df(u_t)e) \leq -\lambda|e|^2$ for *all* t along solution u_t. Otherwise merely indicates potentially more complex dynamics.

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Example: Lorenz'63

Consider

$$\begin{aligned} f_1(u) &= \alpha(u_2 - u_1) \\ f_2(u) &= -\alpha u_1 - u_1 u_3 - u_2 \\ f_3(u) &= u_1 u_2 - \gamma(u_3 - \beta) \end{aligned}$$

with $\alpha = 10, \beta = 38, \gamma = \frac{8}{3}$. The system exhibits

Energy balance Solutions are asymptotically confined to some large energy sphere $|u|^2 \leq K$, but ...

Local instability There are three fixed points ($f(u) = 0$ has three roots), all of which have at least one unstable direction ($Df(u)$ has at least one positive eigenvalue.)

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Probabilistic approach

on the level of distributions rather than solutions

Switch to discrete time (for simplicity)

$$u_{n+1} = f(u_n).$$

Suppose $u_n \overset{\mathcal{D}}{\sim} p_n(x)\mathrm{d}x$. Then $u_{n+1} \overset{\mathcal{D}}{\sim} \mathscr{F}p_n(x)\mathrm{d}x$ where

$$\mathscr{F}\phi(x) := \sum_{\{y;f(y)=x\}} \frac{\phi(y)}{|Df(y)|}.$$

is the *Transfer operator*. (The proof is Exercise 2; need to assume that every image point x has finite number of preimages y, and $|Df(y)| \neq 0$.)

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Application to doubling map

Very chaotic!

Define f through

$$u_{n+1} = 2u_n \bmod 1.$$

Can be interpreted as smooth map on unit circle "doubling the angle":

$$\mathscr{F}\phi(x) = \frac{1}{2}\phi(\frac{x}{2}) + \frac{1}{2}\phi(\frac{x+1}{2})$$

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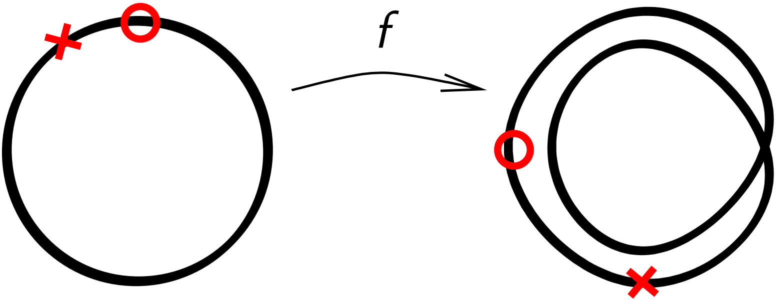

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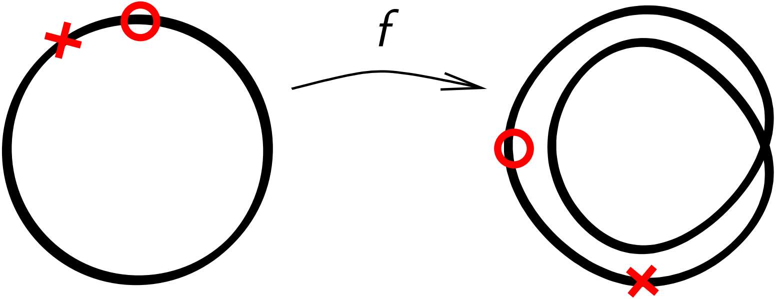

$$\mathscr{F}\phi(x) = \frac{1}{2}\phi(\frac{x}{2}) + \frac{1}{2}\phi(\frac{x+1}{2})$$

$\mathscr{F}$ has smoothing effect!

Thanks to chaos

Assume density p satisfies

$$\frac{p(x)}{p(y)} \leq e^{K|x-y|}$$

for all $x, y \in [0, 1]$. Smaller K $\Leftrightarrow$ density is smoother.

$$\begin{aligned}
\frac{\mathscr{F}p(x)}{\mathscr{F}p(y)} &= \frac{\frac{1}{2}p(\frac{x}{2}) + \frac{1}{2}p(\frac{x+1}{2})}{\frac{1}{2}p(\frac{y}{2}) + \frac{1}{2}p(\frac{y+1}{2})} \\
&\leq \frac{p(\frac{y}{2})e^{K|\frac{x}{2}-\frac{y}{2}|} + p(\frac{y+1}{2})e^{K|\frac{x+1}{2}-\frac{y+1}{2}|}}{p(\frac{y}{2}) + p(\frac{y+1}{2})} \\
&\leq e^{\frac{K}{2}|x-y|}.
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Application of $\mathscr{F}$ renders densities smoother!

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Exercise 3: application to filtering

Consider

$$u_n = 2u_{n-1} \bmod 1 \text{ (dynamics)}, \qquad y_n = u_n + \sigma r_n \text{ (observations)}.$$

with $\{r_1, r_2, \ldots\}$ iid standard normal random variables. The density $p_n(x) := p(u_n = x | y_{1:n})$ satisfies the following recursion:

$$p_n(x) = c g(x, y_n) \mathscr{F} p_{n-1}(x),$$

where $g(x, y) := \exp\left(-\frac{(x-y)^2}{2\sigma^2}\right)$ and c is a normalising constant.

1. Show that the smoothness parameter K_n of p_n satisfies

$$K_n \leq \frac{1}{2} K_{n-1} + L(y_n)$$

   for some function L.

2. Find a *lower* bound on the variance of p_n in terms of the smoothness parameter K_n.

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