Please read these instructions before the test starts. You can also read the question paper before the test starts. You will be allowed to start writing at the formal beginning of the test. You will be given five minutes after the test has finished to complete the front of any answer books used. Please do not remove the question paper from the test room.

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7 Dec 2015 **MTMW12 2015/6**

No calculator necessary

UNIVERSITY OF READING

Introduction to Numerical Modelling (MTMW12)

1 hour and 45 minutes

Answer **ALL** questions.

Total 50 marks.

1. A function f is known at locations x_j, x_{j+1} and x_{j+2} where $x_j = j\Delta x$ for $j = 0, 1, 2, ...$ and Δx is a fixed spacing in the x direction. We will use the notation $f_j = f(x_j)$. Given f_j, f_{j+1} and f_{j+2}, use Taylor series to find an approximation for the second derivative, $f'' = \partial^2 f/\partial x^2$ at location x_j. In other words find f_j'' and show the order of accuracy of this approximation.

[10 marks]

2. This question is about the following finite difference numerical scheme:

$$\frac{\phi_j^{(n+1)} - \phi_j^{(n)}}{\Delta t} + u\frac{\phi_{j+1}^{(n+1)} - \phi_{j-1}^{(n+1)}}{2\Delta x} = 0 \tag{1}$$

where u is the wind speed, ϕ is the dependent variable and ϕ is a function of space and time. $\phi_j^{(n)}$ is defined to be ϕ at position $x_j = j\Delta x$ and at time $n\Delta t$ where Δx is the grid spacing in the x direction and Δt is the time-step.

(a) Write down the partial differential equation that this scheme is designed to solve. What is the name of this equation?

[3 marks]

(b) State if this numerical method is backward, forward or centred in time, backward, forward or centred in space and explicit or implicit.

[3 marks]

(c) Use von-Neumann stability analysis to find the stability limits of this scheme. Throughout your analysis, you can use the Courant number, $c = u\Delta x/\Delta t$ and find the amplification factor, A, for each wavenumber, k.

[10 marks]

3. I am solving the one-dimensional diffusion equation using the forward in time, centred in space finite difference scheme for ϕ:

$$\frac{\phi_j^{(n+1)} - \phi_j^{(n)}}{\Delta t} = K\frac{\phi_{j-1}^{(n)} - 2\phi_j^{(n)} + \phi_{j+1}^{(n)}}{\Delta x^2} \tag{2}$$

where K is the diffusion coefficient, $\phi_j^{(n)}$ is ϕ at position $x_j = j\Delta x$ and at time $n\Delta t$ where Δx is the spacing in the x direction and Δt is the time-step. My code uses single precision floating point numbers. The domain is between $x = 0$ m and $x = 1$ m using $n_x + 1$ points with boundary conditions $\phi_0 = 0$ and $\phi_{n_x} = 0$. My initial conditions are:

$$\phi(x,0) = \begin{cases} 1 & 0.4 \le x \le 0.6 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

I initialise my model so that:

$$\phi_j^{(0)} = \int_{x_j-\frac{1}{2}\Delta x}^{x_j+\frac{1}{2}\Delta x} \phi(x,0)\,dx \ \text{ for } j = 1, 2, \cdots n_x - 1. \tag{4}$$

I compare my results with an analytic solution for the diffusion equation starting from the same initial conditions but which assumes an infinite domain. I use $K = 10^{-3}\text{m}^2\text{s}^{-1}$ and I do a number of simulations with different Δt, Δx and n_t (total number of time-steps) as shown in table 1, some of which have $n_t\Delta t = T = 4$ seconds and some $T = 100$ seconds. When I use $\Delta x = 0.1$ m I get the results shown in figure 1. For each of the run lengths I plot the ℓ_2 error norm as a function of Δx and I get the results in figure 2. Explain the curves in figure 2.

[12 marks]

Δx (m)	Δt (s)	n_t ($T = 4$s)	n_t ($T = 100$s)
$0{\cdot}1$	2	2	50
$0{\cdot}05$	$0{\cdot}5$	8	200
$0{\cdot}025$	$0{\cdot}125$	32	800
$0{\cdot}0125$	$0{\cdot}03125$	128	3200
$0{\cdot}01$	$0{\cdot}02$	200	5000

Table 1: Resolution, time-steps and number of time-steps for the numerical solutions of the diffusion equation.

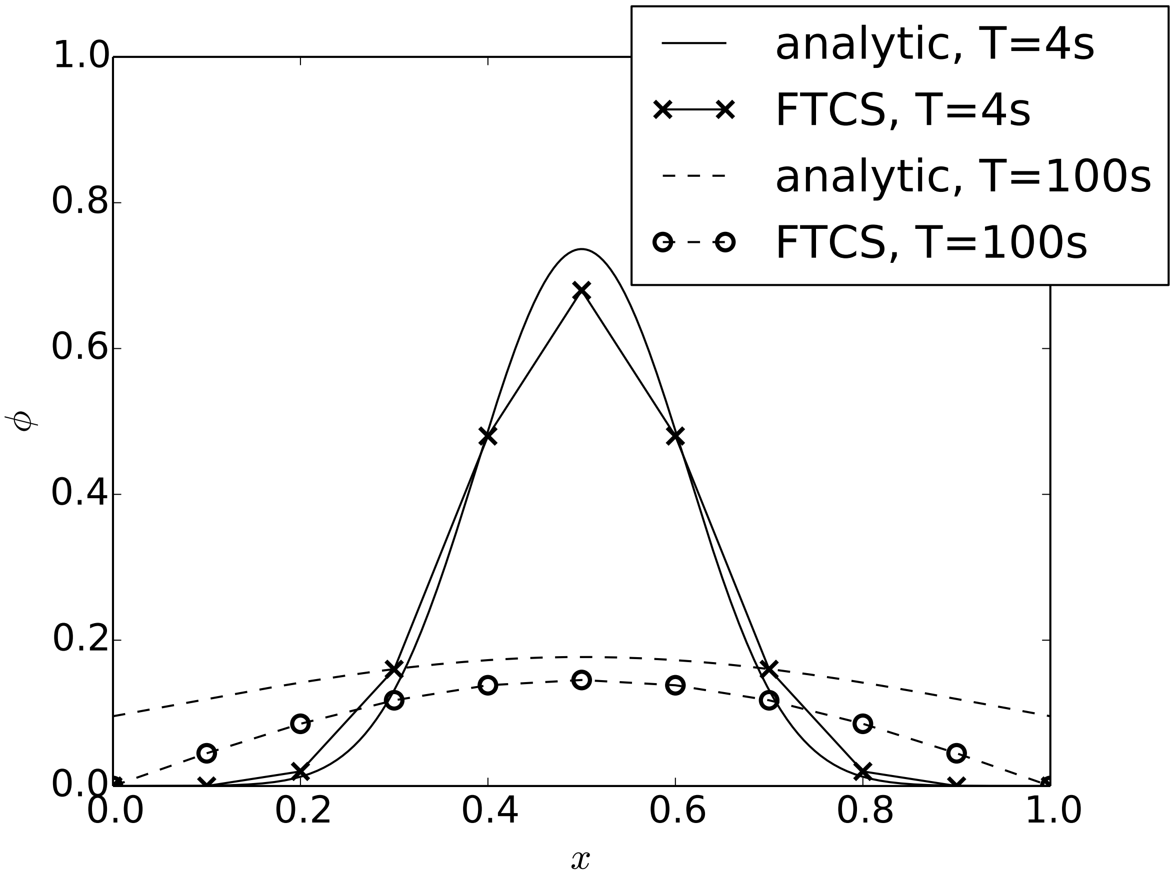

Figure 1: Numerical and analytic solutions of the diffusion equation using $\Delta x = 0.1$ m and $\Delta t = 2$ s.

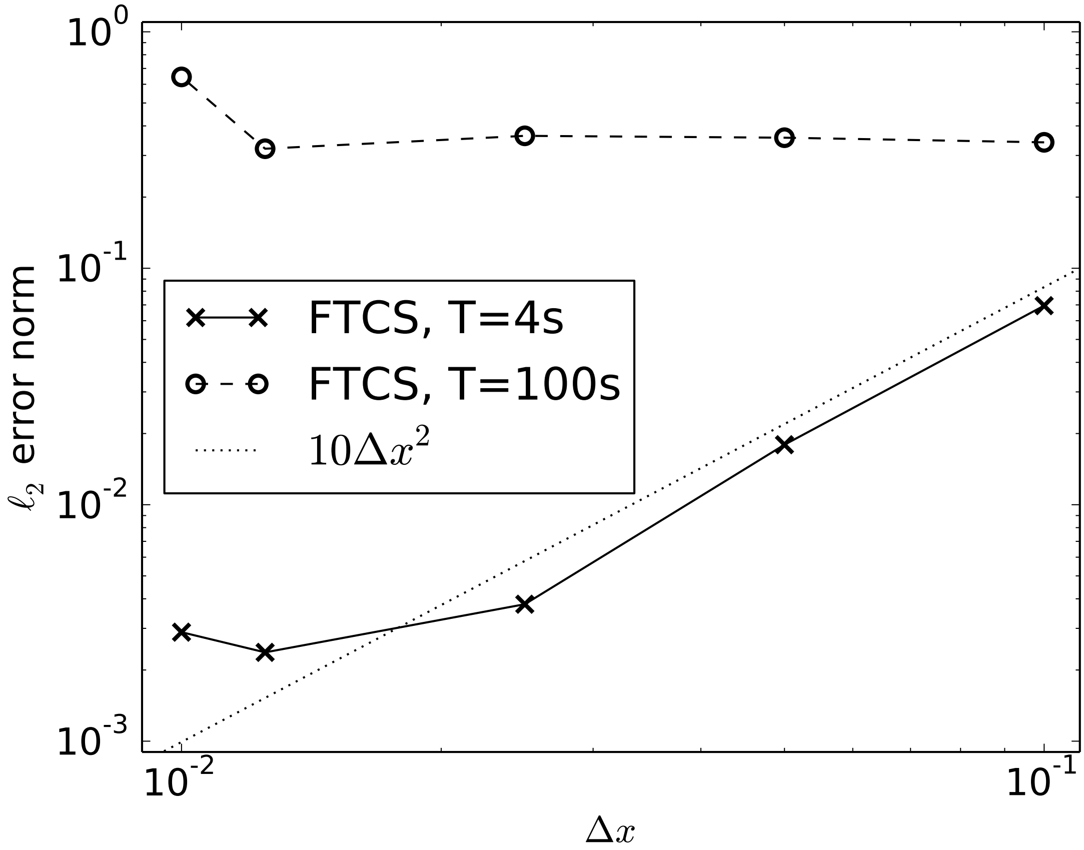

Figure 2: ℓ_2 error norms for the numerical solution of the diffusion equation with parameters defined in table 1.

4. Figure 3 is adapted from Skamarock (2004, Monthly Weather Review "Evaluating mesoscale NWP models using kinetic energy spectra") and shows the power spectra of the kinetic energy of three versions of the WRF model. Version 1 has no artificial diffusion, version 2 has artificial diffusion of the form ∇^2 and version 3 has artificial hyper-diffusion of the form ∇^4.

   (a) State which line (a, b and c) in figure 3 refers to which model version (1, 2 or 3)

   [3 marks]

   (b) Referring to the figure, explain the effects of diffusion and hyper-diffusion and thus why you made your choice above.

   [9 marks]

[End of Question Paper]

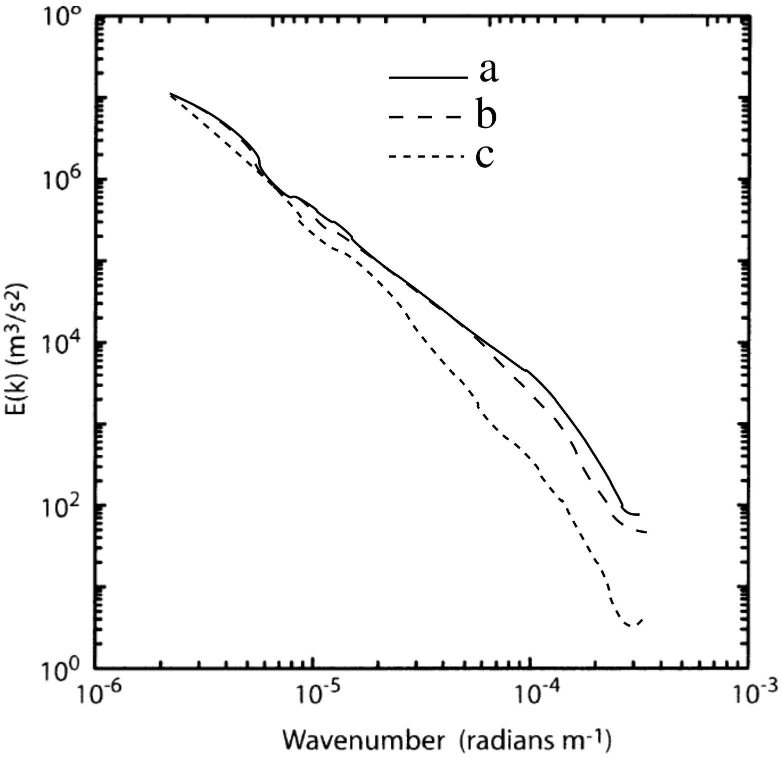

Figure 3: Power spectra of the kinetic energy of the WRF model adapted from Skamarock (2004, Monthly Weather Review “Evaluating mesoscale NWP models using kinetic energy spectra”)