On admission to the examination room, you should acquaint yourself with the instructions below. You must listen carefully to all instructions given by the invigilators. You may read the question paper, but must not write anything until the invigilator informs you that you may start the examination.

You will be given five minutes at the end of the examination to complete the front of any answer books used.

DO NOT REMOVE THE QUESTION PAPER FROM THE EXAM ROOM.

April 2016 **MTMD02 2015/16 A 001**

Answer Book
Any non-programmable calculator permitted
Any bilingual English language dictionary permitted
Lecture notes permitted

UNIVERSITY OF READING

Operational data assimilation techniques (MTMD02)

Two hours

Answer **ANY TWO** questions.

The marks for the individual components of each question are given in [] brackets. The total mark for the paper is 100.

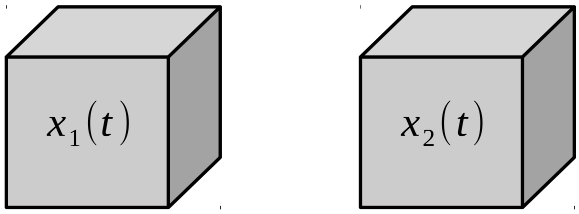

Figure 1: Two air parcels and the environment.

1. Two air parcels have temperatures $x_1(t)$ and $x_2(t)$ respectively (Fig. 1). These time-dependent temperatures are to be estimated by data assimilation. A model for how the state $\mathbf{x}(t)$ evolves is the following Newtonian cooling model:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} T_{\text{env}} + (x_1(0) - T_{\text{env}})e^{-\alpha_1 t} \\ T_{\text{env}} + (x_2(0) - T_{\text{env}})e^{-\alpha_2 t} \end{pmatrix},$$

where T_{env} is the (fixed) environmental temperature and α_1 and α_2 are the parcels' respective cooling rates (T_{env}, α_1 and α_2 constant and are known perfectly).

(a) Make a sketch of the behaviour of $x_1(t)$ and $x_2(t)$ according to this model when $x_1(0) > x_2(0) > T_{\text{env}}$ and $\alpha_1 > \alpha_2$.

[5 marks]

A temperature observation of air mass 1 is made at each of the times t_1 and t_2 ($y_1(t_1)$ and $y_1(t_2)$ respectively) where $t_1, t_2 \geq 0$. Observation errors are mutually uncorrelated and have standard deviations σ_1 and σ_2 respectively. A background state is also provided, $\mathbf{x}^{\text{B}}(0)$, where background errors for each air mass temperature are mutually uncorrelated and each has standard deviation σ_{B}. The observations and background are to be combined using strong constraint 4D-Var to give a new estimate of $\mathbf{x}(0)$.

(b) What is meant by the terms ”4D-Var” and ”strong constraint”?

[5 marks]

(c) With reference to the information given, write down:

(i) the observation error covariance matrix at t_1 ($\mathbf{R}_1$) and at t_2 ($\mathbf{R}_2$),

[3 marks]

(ii) and the background error covariance matrix $\mathbf{B}$ at $t = 0$.

[2 marks]

Let $x_1(0)$ and $x_2(0)$ be the control variables of the strong-constraint 4D-Var.

(d) Write down forms of the model's version of the two observations (call y_1^{m} and y_2^{m}) 4D-Var.

[4 marks]

(e) Are these observation operators linear or non-linear functions of the control variables?

[2 marks]

(f) Write down the 4D-Var cost function for this case, in terms of $x_1(0)$, $x_2(0)$, $x_1^{\mathrm{B}}(0)$, $x_2^{\mathrm{B}}(0)$, $y_1(t_1)$, $y_1(t_2)$, the covariance matrices from (c) and the observation operators from (d).

[4 marks]

(g) Write down the derivative of the cost function with respect to $x_1(0)$ and $x_2(0)$ by either differentiating it by hand or by applying an appropriate standard formula for the gradient.

[6 marks]

(h) Find the analysis state that minimizes the cost function.

[10 marks]

Suppose that observation 1 is much more precise than both the background state and observation 2.

(i) Write down this statement mathematically.

[2 marks]

(j) Show that in this case the analysis of $x_1(0)$ and $x_2(0)$ approximate to:

$$\begin{aligned} x_1^{\mathrm{A}}(0) &= (y_1 - T_{\mathrm{env}}) \exp(\alpha_1 t_1) + T_{\mathrm{env}}, \\ x_2^{\mathrm{A}}(0) &= x_2^{\mathrm{B}}(0). \end{aligned}$$

[3 marks]

(k) In the light of the behaviour of the model in your sketch from part (a), comment on the presence of the exponentially increasing part $\exp(\alpha_1 t_1)$ in the above result for the analysis $x_1^{\mathrm{A}}(0)$. Why does this poses a potential difficulty for the assimilation of observations made at distant times?

[4 marks]

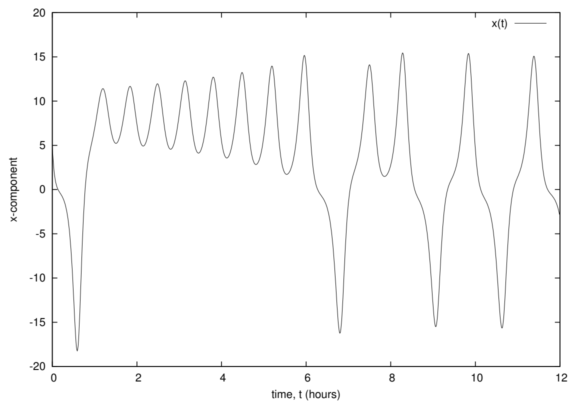

Figure 2: Sample run of the Lorenz model.

2. A phenomenon is being studied which is represented by the state vector $\mathbf{x}$, and which comprises three-variables: $\mathbf{x} = (\ x \ \ y \ \ z\ )^{\mathrm{T}}$ (when you write your answers to this question you should distinguish between $\mathbf{x}$ and x by underlining vectors). The propagation in time of $\mathbf{x}$ is to be modelled with the Lorenz model:

$$\begin{aligned} \frac{dx}{dt} &= -\sigma(x-y), \\ \frac{dy}{dt} &= \rho x - y - xz, \\ \frac{dz}{dt} &= xy - \beta z, \end{aligned}$$

where t is time, and σ, ρ and β are known positive constants. A sample run of this model over 12 hours is shown in Fig. 2. An observation vector, $\mathbf{y}(t) = (\ y_1 \ \ y_2 \ \ y_3 \ \ y_4\ )^{\mathrm{T}}$, observing x, y, z and $x^2 + y^2$ respectively, is made at each hour throughout a 12-hour asimilation window.

(a) Give pros and cons of applying the following data assimilation methods to this model with a 12-hour assimilation window: (i) Kalman filter, (ii) 3D-Var, (iii) strong constraint 4D-Var, (iv) weak constraint 4D-Var and (v) the EnKF. The number of pros plus the number of cons should equal four per method and some methods may share the same pros/cons. Your answer must include one point per method concerning which observations over the 12-hour window can accurately be assimilated.

[20 marks]

For the rest of this question it is decided to use a 1-hour assimilation window instead of a 12-hour window, where observations made for the analysis at hour t are assimilated in the time window defined by τ: $t \leq \tau < t+1$.

(b) For each assimilation method listed in (a), state which pros and cons no longer apply and why.

[5 marks]

(c) Consider the strong constraint 4D-Var method and let the start of the 1-hour assimilation window under examination be at $t = 0$.

(i) Write down the full observation operator vector that applies, giving the model's version of the observations, $\mathbf{y}^{\mathrm{m}}(t) = \boldsymbol{\mathcal{H}}\left(\boldsymbol{\mathcal{M}}_t(\mathbf{x}(0))\right)$, for this assimilation window, where $\boldsymbol{\mathcal{H}}$ is the observation operator, and $\mathbf{x}(t) = \boldsymbol{\mathcal{M}}_t(\mathbf{x}(0))$. You should include the $\boldsymbol{\mathcal{M}}_t$ step only if it is needed for this particular application of the 4D-Var equations.

[7 marks]

For incremental 4D-Var, $\mathbf{y}^{\mathrm{m}}(t)$ may be written in the form $\mathbf{y}^{\mathrm{m}}(t) = \boldsymbol{\mathcal{H}}\left(\boldsymbol{\mathcal{M}}_t(\mathbf{x}^{\mathrm{B}}(0))\right) + \mathbf{H}_t\mathrm{M}_t\delta\mathbf{x}(t)$, where $\mathbf{H}_t$ and M_t are the Jacobians of $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{M}}_t$ respectively (linearized about $\mathbf{x}^{\mathrm{B}}(0)$), where $\mathrm{M}_0 = \mathbf{I}$.

(ii) How many rows and how many columns does $\mathbf{H}_t\mathrm{M}_t$ have?

[2 marks]

(iii) Write down the matrix elements of $\mathbf{H}_t\mathrm{M}_t$ for this case.

[3 marks]

(iv) Write down the observation term of the incremental strong-constraint 4D-Var cost function for this particular time window. Let $\mathbf{x}^{\mathrm{B}}(0) = \begin{pmatrix} x^{\mathrm{B}}(0) & y^{\mathrm{B}}(0) & z^{\mathrm{B}}(0) \end{pmatrix}^{\mathrm{T}}$ be the background state, $\mathbf{B}$ be its error covariance matrix, and σ_1, σ_2, σ_3, and σ_4 be the standard deviations of the respective observation errors. For the purposes of this question you may assume that observations are uncorrelated. You do not need to expand out the vectors/matrices, but you do need to show their elements.

[8 marks]

(d) A different phenomenon is to be studied which, instead of 3, requires 10^8 variables in $\mathbf{x}$. Discuss any extra issues of applying each assimilation method listed in (a).

[5 marks]

3. (a) Consider the dynamical system:

$$\begin{pmatrix} u(t+1) \\ v(t+1) \\ w(t+1) \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix},$$

and an assimilation window that includes times $t = 0$, 1 and 2. At each of these times two observations are made: one of $u(t)$ and another of $u(t) + v(t)$. Write down the matrix operator that gives the model's version of the observations for:

(i) time $t = 0$,

[2 marks]

(ii) and for the whole assimilation window (i.e. the observability matrix).

[4 marks]

(iii) Even though there are only 3 variables, but 6 observations in total for this window, show mathematically that the system is not fully observed.

[6 marks]

(iv) Suggest a way of fully observing this system.

[2 marks]

(b) With reference to ensemble data assimilation methods, explain what is meant by filter divergence.

[2 marks]

(c) An ensemble Kalman filter (EnKF) with 100 ensemble members is applied to a weather forecasting problem. The observation increments are affecting regions geographically located a long way from the observation sites. A meteorologist tells you that this does not seem to be realistic.

(i) Describe why the unphysical assimilation increments may be occurring.

[8 marks]

(ii) Describe the technique that is commonly used to reduce the unphysical assimilation increments, while still allowing all observations to be assimilated at once, and not using more than 100 members. Comment on its mathematical properties. (You should describe the key ideas of the technique but you do not need to give the full implementation of the EnKF).

[10 marks]

(d) Several operational weather forecasting centres are now using hybrid ensemble-variational techniques for data assimilation. For example the cost function for Hybrid-En-3D-Var is:

$$J(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{x}^{\mathrm{B}})^{\mathrm{T}}\mathbf{P}^{\mathrm{H}^{-1}}(\mathbf{x} - \mathbf{x}^{\mathrm{B}}) + \frac{1}{2}(\mathbf{y} - \mathbf{H}\mathbf{x})^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y} - \mathbf{H}\mathbf{x}),$$

where all symbols are standard, and $\mathbf{P}^{\mathrm{H}}$ is the hybrid background error covariance matrix.

(i) How is $\mathbf{P}^{\mathrm{H}}$ constructed in terms of the background error covariance matrix used in pure 3D-Var and that used in the pure EnKF.

[4 marks]

(ii) Compare and contrast the hybrid approach to the pure 3D-Var and pure EnKF methods.

[12 marks]

[End of Question Paper]