On admission to the examination room, you should acquaint yourself with the instructions below. You must listen carefully to all instructions given by the invigilators. You may read the question paper, but must not write anything until the invigilator informs you that you may start the examination.

You will be given five minutes at the end of the examination to complete the front of any answer books used.

DO NOT REMOVE THIS QUESTION PAPER FROM THE EXAM ROOM.

April 2016 **MAMNSP 2015 / 16 A001**

Answer Book
Lecture notes permitted
Any bilingual English language dictionary permitted
Any non-programmable calculator is permitted

UNIVERSITY OF READING

Numerical Solutions of Partial Differential Equations (MAMNSP)

Two hours

Answer **ALL QUESTIONS**

The marks for the individual components of each question are given in [] brackets. The total mark for the paper is 100. This paper is worth 80% of the total module mark.

1. Let $\Omega = (0,1) \times (0,1)$, denote the boundary of Ω by Γ, and set

$$\Gamma_N := \{(x,y) : x = 1, 0 < y < 1\},$$

and $\Gamma_D := \Gamma \backslash \Gamma_N$. Consider the elliptic boundary-value problem: find $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

$$\begin{aligned} -\Delta u + \alpha^2 u = f, &\quad \text{in } \Omega, \\ u = 0, &\quad \text{on } \Gamma_D, \\ \tfrac{\partial u}{\partial x} = 0, &\quad \text{on } \Gamma_N, \end{aligned}$$

where $\alpha \geq 0$ and $f \in C^2(\Omega)$.

(a) Construct a five-point difference scheme for the approximate solution of this problem on a uniform mesh. Rewrite the difference scheme as a system of linear equations, $AU = F$ (where you should define A, U and F) and comment on the structure of the matrix A.

[15 marks]

(b) Explain what is meant by the truncation error of this finite difference scheme, and derive as sharp a bound as you can on the truncation error.

[15 marks]

(c) For the case that

$$f(x,y) = \begin{cases} 0, & x \leq y, \\ 1, & x > y, \end{cases}$$

(noting that in this case $f \notin C(\Omega)$) explain why the finite difference method described above may not be appropriate. Give brief details of how the finite volume method could be applied in this case, with justification for why it may be preferable.

[10 marks]

2. (a) Consider the solution of

$$-u''(x) = g(x), \quad x \in (0,1), \tag{1}$$
$$u(0) = 0, \quad u'(1) = 1, \tag{2}$$

where $g \in L^2(0,1)$.

(i) Show that (1)–(2) may be given the weak formulation:

$$\text{Find } u \in V \text{ such that } a(u,v) = l(v) \text{ for all } v \in V, \tag{3}$$

where $V := \{v \in H^1(0,1) : v(0) = 0\}$ and for all $v, w \in V$,

$$\begin{aligned} a(v,w) &:= \int_0^1 v'(x)w'(x)\,\mathrm{d}x, \\ l(v) &:= \int_0^1 g(x)v(x)\,\mathrm{d}x + v(1). \end{aligned}$$

[5 marks]

(ii) By writing

$$v(x) = v(1) - \int_x^1 v'(\xi)\,\mathrm{d}\xi,$$

or otherwise, show that

$$|v(1)| \leq \sqrt{2}\|v\|_{H^1(0,1)},$$

for all $v \in V$.

[5 marks]

(iii) Show that (3) has a unique solution $u \in V$, stating clearly any theorems you may use, and derive a stability bound on $\|u\|_{H^1(0,1)}$.

[15 marks]

(b) Consider the solution of

$$-\Delta u = f, \quad \text{in } \Omega, \tag{4}$$
$$u = 0, \quad \text{on } \Gamma, \tag{5}$$

where $\Omega = (0,1) \times (0,1)$, Γ is the boundary of Ω, and $f \in L^2(\Omega)$. Show that (4)–(5) may be given the weak formulation:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u,v) = l(v) \text{ for all } v \in H_0^1(\Omega),$$

where for all $v, w \in H^1(\Omega)$,

$$\begin{aligned} a(v,w) &:= \int_\Omega \nabla v . \nabla w \, \mathrm{d}\Omega, \\ l(v) &:= \int_\Omega f v \, \mathrm{d}\Omega, \end{aligned}$$

and describe how one may go about constructing a finite element approximation to u, stating (with justification) specific choices of approximation space and basis functions.

[15 marks]

3. Consider the scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{\Delta x} = 0$$

for approximating the solution of the linear wave equation

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0.$$

Here $u_j^n \approx u(x_j, t_n)$, where x_j and t_n, $j, n = 0, 1, 2, \ldots$, are, respectively, the spatial and temporal nodes of a space-time mesh, with Δx the space step and Δt the time step.

(a) Show that the scheme is first order accurate in time and second order accurate in space, but unstable.

[10 marks]

(b) Show that if the extra term

$$2\frac{\Delta t}{\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n),$$

is added to the right hand side, the resulting scheme is conditionally stable, and find the condition.

[10 marks]

[End of Question Paper]