On admission to the examination room, you should acquaint yourself with the instructions below. You must listen carefully to all instructions given by the invigilators. You may read the question paper, but must not write anything until the invigilator informs you that you may start the examination.

You will be given five minutes at the end of the examination to complete the front of any answer books used.

DO NOT REMOVE THIS QUESTION PAPER FROM THE EXAM ROOM.

January 2016 **MAMNSO 2015 / 16 A001**

Answer Book
Lecture notes permitted
Any bilingual English language dictionary permitted
Any non-programmable calculator is permitted

UNIVERSITY OF READING

Numerical Solutions of Ordinary Differential Equations (MAMNSO)

Three hours

Answer **ALL QUESTIONS**

The marks for the individual components of each question are given in [] brackets. The total mark for the paper is 100. This paper is worth 80% of the total module mark.

1. Consider an implicit linear multistep method

$$y_{n+2} - y_n = h \sum_{k=0}^{2} \beta_k f_{n+k}$$

where h is the step size.

(a) Find the values of the coefficients $\beta_k,\ k = 0, 1, 2$, for the method to approximate the solution of the differential equation $y' = f(t, y)$ with the highest possible order of consistency and give the error constant for the obtained method.

[10 marks]

(b) Deduce whether the obtained highest order method is convergent.

[5 marks]

(c) Find the interval of absolute stability of this method with the highest order, if there is any.

[10 marks]

2. Consider the s-stage Runge-Kutta (RK) methods

$$\begin{aligned} y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i f(t_n + c_i h, \xi_i), \\ \xi_i &= y_n + h \sum_{j=1}^{s} a_{i,j} f(t_n + c_j h, \xi_j), i = 1, \ldots, s \end{aligned}$$

where $\{a_{i,j}\}$, $\{b_i\}$ and $\{c_i\}$, $i, j = 1, \ldots, s$, are constants, for approximating the solution to the differential equation $y' = f(t, y)$.

(a) Show that for the s-stage RK method to be consistent of order $p \geq 1$ when applied to approximate the solution of the initial value problem

$$\begin{cases} y'(t) = 1,\ t > 0 \\ y(0) = 0, \end{cases}$$

the coefficients must satisfy

$$\sum_{i=1}^{s} b_i = 1 \quad \text{and} \quad c_i = \sum_{j]1}^{s} a_{i,j}, \quad i = 1, \ldots, s$$

[8 marks]

(b) Show that the RK method with the Butcher array

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & a & a \\ \hline & b & b \end{array}$$

is equivalent to the Trapezoidal Rule.

[6 marks]

(c) Consider a 2-stage *explicit* RK method

$$y_{n+1} = y_n + h\left(\frac{2}{5}f(t_n, y_n) + \theta f\left(t_n + ah, y_n + ahf(t_n, y_n)\right)\right).$$

(i) Find the values of θ and a such that this method is consistent of order $p = 2$.

[5 marks]

(ii) Find the range of step size for the order 2 method obtained in (i) to be absolutely stable when applied to integrate the linear system of order differential equations

$$\begin{aligned} u'(t) &= -9u(t) - 2v(t) \\ v'(t) &= 4u(t) - 3v(t). \end{aligned}$$

Note that you can use the interval of absolute stability of 2-stage Runge-Kutta method without proof.

[6 marks]

3. (a) Discretise the boundary value problem (BVP)

$$\begin{cases} -y''(x) + (3 + 5\sin(\pi x))y'(x) + e^x y(x) = 0, & x \in (0, 1) \\ y(0) = 0, \; y(1) = 1 \end{cases}$$

using a three-term finite difference approximation on a uniform mesh of step size $h = \frac{1}{N}$ and give the coefficient matrix of the difference equation.

[7 marks]

(b) Find the minimum number of subintervals N which is a sufficient condition for the coefficient matrix in (a) to be non-singular.

[8 marks]

(c) Use the maximum principle with auxiliary function $g(x) = \frac{x}{3}$ to show that the error $e_i = y(x_i) - y_i$ between the true solution and the finite difference approximation is $\mathcal{O}(h^2)$.

Note that you can use without proof the upper bound of the truncation error $T_j \leq \frac{1}{12}Mh^2, j = 0, 1, \ldots, N$, *with* M *a positive constant.*

[10 marks]

4. Consider the one-step numerical method

$$y_{n+1} = y_n + h\left[(1-\theta)f_n + \theta f_{n+1}\right], \theta \in [0, 1]$$

for approximating the solution of the ordinary differential equation $y'(t) = f(t, y(t))$.

(a) Find whether the method with $\theta = 0$ can approximate the solution $y(t) = \left(\frac{3t}{4}\right)^{4/3}$ of the initial value problem (IVP)

$$y'(t) = y^{1/4}, \quad y(0) = 0.$$

Relate your finding to the well-posedness condition of the IVP.

[7 marks]

(b) Find the region of absolute stability for the methods with $\theta \in [0, 1/2)$.

[10 marks]

(c) Consider applying the method of $\theta = 1/2$ to solve the initial value problem

$$y'(t) = 1 + y^2(t), \quad y(0) = 0.$$

Show that it is necessary to solve a quadratic equation in order to determine y_{n+1} from y_n and that an appropriate root can be identified by making use of the property that $y_{n+1} \to y_n$ as $h \to 0$.

[8 marks]

[End of Question Paper]