On admission to the examination room, you should acquaint yourself with the instructions below. You <u>must</u> listen carefully to all instructions given by the invigilators. You may read the question paper, but must <u>not</u> write anything until the invigilator informs you that you may start the examination.

You will be given five minutes at the end of the examination to complete the front of any answer books used.

DO NOT REMOVE THIS QUESTION PAPER FROM THE EXAM ROOM

January 2016 **MAMB10/ 2015/16 A001**

Any non-programmable calculator permitted
Lecture notes permitted

UNIVERSITY OF READING

Theory and techniques of data assimilation

2 hours

Answer all questions
60 marks are available on this paper. This paper is worth 80% of the total module mark.

1. Let a model state $\mathbf{x}$ consist of two variables r, θ defined at a single point, so that $\mathbf{x} = (r, \theta)^T$. Assume that we have an unbiased background estimate of the state $\mathbf{x}^b = (r^b, \theta^b)^T$, with error covariance matrix

$$\mathbf{B} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{pmatrix}.$$

Suppose that we make an observation y of the quantity $r^2\theta$, with error variance σ_o^2 and that we assimilate this observation using the standard formula for the best linear unbiased estimate (BLUE).

(a) Define the observation operator for this problem and find its linearization. Hence find the analysis vector $\mathbf{x}^a = (r^a, \theta^a)^T$.

[12 marks]

(b) Suppose now that the observation is biased, with the expected value of its error equal to a non-zero constant c. Assuming the tangent linear approximation where necessary, find the expected value of the analysis error in r.

[8 marks]

2. Let a model state $\mathbf{x}$ consist of three variables u, v, h at a single point, so that $\mathbf{x} = (u, v, h)^T$, and suppose that the evolution of the variables is given by the linear model equations

$$\begin{aligned} u_{n+1} &= u_n + \alpha \Delta t h_n, \\ v_{n+1} &= v_n + \beta \Delta t h_n, \\ h_{n+1} &= h_n + \gamma \Delta t^2 v_n, \end{aligned}$$

where α, β, γ are scalar constants, n is the time level and Δt is the model time step.

Now suppose that we have a background state $\mathbf{x}^b = (u^b, v^b, h^b)^T$ at time t_b, with error covariance matrix

$$\mathbf{B} = \begin{pmatrix} \sigma_u^2 & 0 & 0 \\ 0 & \sigma_v^2 & 0 \\ 0 & 0 & \sigma_h^2 \end{pmatrix}.$$

Let $\tilde{v}$ be an observation of v at time $t_b + \Delta t$, with error variance σ_o^2. We perform a 4D-Var analysis over the time window $[t_b, t_b + \Delta t]$.

(a) Assuming that the initial guess is equal to the background, find the gradient of the 4D-Var cost function on the first iteration, expressing any values of the background field in terms of their time level t_b values.

[9 marks]

(b) Calculate the inverse of the analysis error covariance matrix by means of the Hessian information. [You do not need to invert the matrix.]

[5 marks]

(c) Calculate what the inverse of the analysis error covariance matrix would be if the observation $\tilde{v}$ was at time t_b. Comment on the fact that this matrix is diagonal, whereas the matrix calculated in part (b) is not.

[6 marks]

3. Suppose we have a dynamical system for a vector $\mathbf{x} = (v, w)^T$ where v and w are scalar quantities. Let the dynamical system be represented by the discrete equations

$$\begin{aligned} v_{n+1} &= v_n + 3w_n, \\ w_{n+1} &= w_n, \end{aligned}$$

where the subscript indicates the time level. Assume this model is perfect, i.e. there is no model error. We wish to determine the state of the system using a Kalman filter using observations of v only.

(a) Suppose that we have observations $\tilde{v}_0, \tilde{v}_1$ of v at times t_0, t_1 respectively, each with error variance σ_o^2. We apply a Kalman filter starting from a background state $\mathbf{x}^b = (v^b, w^b)^T$ at time t_0 with error covariance matrix

$$\mathbf{P}^f(t_0) = \begin{pmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_w^2 \end{pmatrix}.$$

The analysis $\mathbf{x}^a(t_0)$ at time t_0 is given by

$$\begin{pmatrix} v^a(t_0) \\ w^a(t_0) \end{pmatrix} = \begin{pmatrix} v^b + \alpha(\tilde{v}_0 - v^b) \\ w^b \end{pmatrix}$$

where $\alpha = \sigma_v^2/(\sigma_v^2 + \sigma_o^2)$, with analysis error covariance matrix

$$\mathbf{P}^a(t_0) = \begin{pmatrix} \alpha\sigma_o^2 & 0 \\ 0 & \sigma_w^2 \end{pmatrix}.$$

(i) Find the forecast state $\mathbf{x}_1^f$ and its error covariance matrix $\mathbf{P}^f(t_1)$ at time t_1.

[6 marks]

(ii) Hence find the analysis $(v(t_1), w(t_1))^T$ at time t_1, expressing all variables in terms of their time level t_0 values.

[10 marks]

(b) If we applied a 4D-Var assimilation over the time window $[t_0, t_1]$, would you expect the analysis $\mathbf{x}^a(t_0)$ at the initial time t_0 to have the same, better or worse accuracy than the Kalman filter analysis at the same time? Give reasons for your answer

[4 marks]

[End of Question Paper]