On admission to the examination room, you should acquaint yourself with the instructions below. You must listen carefully to all instructions given by the invigilators. You may read the question paper, but must not write anything until the invigilator informs you that you may start the examination.

You will be given five minutes at the end of the examination to complete the front of any answer books used.

DO NOT REMOVE THIS QUESTION PAPER FROM THE EXAM ROOM

April 2016 **MAMA14 2015/16 A 001**

Lecture notes are permitted
Any bilingual English language dictionary permitted
Any non-programmable calculator permitted

UNIVERSITY OF READING

APPLIED STOCHASTIC PROCESSES (MAMA14 2015/16)

Two hours

Answer **ALL** questions. The marks for the individual components of each question are given in [] brackets. The total mark for the paper is 50. This paper is worth 70% of the total module mark.

1. Suppose that the life time X has density p given by

$$p(x) = c(a - x), \quad \text{for } 0 \le x \le a$$

and $p(x) = 0$ else, where a is a positive constant.

Calculate c, the mean and variance of the life time.

[6 marks]

2. Let X_1, X_2 be independent exponentially distributed random variables with parameters λ_1, λ_2 so that

$$\mathbb{P}(X_i > t) = \exp[-\lambda_i t], \quad \text{for } t \ge 0, i \in \{1, 2\} .$$

Define $N = 1$ if $X_1 < X_2$, otherwise $N = 2$.

(a) Let $U = \min(X_1, X_2)$. Show that $\mathbb{P}(U > t) = \exp[-(\lambda_1 + \lambda_2)t]$ for $t > 0$.

[3 marks]

(b) Show that $\mathbb{P}(N = 1) = \lambda_1/(\lambda_1 + \lambda_2)$.

[4 marks]

(c) Show that $\mathbb{P}(W > t|N = 1) = \exp[-\lambda_2 t]$ where $W = |X_1 - X_2|$.

[5 marks]

3. Let W_t be a standard one dimensional Wiener process.

(a) Define the stochastic process Z_t by $Z_t = W_t - tW_1, t \in [0, 1]$. Show that Z_t is a Gaussian process with

$$\mathbb{E}[Z_t] = 0, \quad \mathbb{E}[Z_t Z_s] = \min(t, s) - ts.$$

[5 marks]

(b) Calculate the mean $\mathbb{E}[Y_t]$ and the correlation function $C(t,s) = \mathbb{E}[Y_t Y_s]$ of the process

$$Y_t = \int_0^t W_s \mathrm{d}s$$

[7 marks]

4. Two cells, A and B, contain together N molecules of gas. At each time instant a molecule is chosen at random from the N, and if the chosen molecule is from cell A it is put into cell B, and if it is from cell B it is put into cell A.

(a) Find the one-step transition probabilities of this Markov chain, where the random variable X_n denotes the number of molecules in cell A at time instant n.

[5 marks]

(b) Define the generating function $G_n(k) = \sum_{j=0}^{N} \mathbb{P}(X_n = j)e^{ijk}$. Show that for $n \to \infty$ this function satisfies

$$0 = (e^{ik} - 1)G_\infty(k) + \frac{1}{iN}(e^{-ik} - e^{ik})G'_\infty(k)$$

where $G'_\infty(k) = \frac{\mathrm{d}}{\mathrm{d}k}G_\infty(k)$.

[10 marks]

(c) Derive the Gaussian approximation by solving the equation for G_∞ up to second order in k for $k \to 0$. Find the corresponding stationary values for the mean and variance.

[5 marks]

[End of Question Paper]