On admission to the examination room, you should acquaint yourself with the instructions below. You must listen carefully to all instructions given by the invigilators. You may read the question paper, but must not write anything until the invigilator informs you that you may start the examination.

You will be given five minutes at the end of the examination to complete the front of any answer books used.

DO NOT REMOVE THE QUESTION PAPER FROM THE EXAM ROOM.

April 2015 **MTMD02/ 2014/15 A001**

Answer Book
Any non-programmable calculator permitted
Any bilingual English language dictionary permitted
Lecture notes permitted

UNIVERSITY OF READING

Operational data assimilation techniques (MTMD02)

Two hours

Answer **ANY TWO** questions.

The marks for the individual components of each question are given in [] brackets. The total mark for the paper is 100.

1. An n-variable 3-D variational data assimilation system is used with p observations. All observations are related to the state space with linear relationships.

   (a) Using the standard vector and matrix notation (as used in the lectures), write down the expression for the analysis increment for this system. For each symbol in this expression, describe its function and state whether it is a scalar, vector or matrix. In addition, for vectors state the number of elements and for matrices state the dimensions (rows $\times$ columns).

   [12 marks]

   (b) A system comprises a state vector with two variables, x_1 and x_2, and there is one observation, y_1, whose observation operator is $ax_1 + bx_2$.

   (i) What are the dimensions of the observation operator matrix (rows $\times$ columns) for this case?

   [2 marks]

   (ii) Write down the observation operator matrix, $\mathbf{H}$.

   [1 marks]

   (iii) Give an example of an observation operator that may have this form and briefly explain the meanings of a and b for your example.

   [3 marks]

   (iv) Use your answer to (a) to give the analysis increment separately for each of the components of the state vector. You may take the $\mathbf{B}$- and $\mathbf{R}$-matrices as:

$$\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} R_{11} \end{pmatrix}.$$

   [6 marks]

   (c) (i) Write down the matrix expression for the optimal analysis error covariance matrix, $\mathbf{A}$, using the standard notation.

[3 marks]

(ii) What part of **A** makes up the analysis error variances and what is their statistical interpretation?

[4 marks]

(iii) Evaluate this matrix for the same system as in (b) (you do not need to expand-out any brackets if you do not wish to).

[7 marks]

(iv) In general what would be the key characteristic of **A** that would indicate that analysis errors are correlated?

[2 marks]

(d) In the same system as in (b), consider the case when background errors between the two components are uncorrelated.

(i) Write down the simplified version of **A** for this case and demonstrate that analysis errors between the two components are correlated.

[4 marks]

(ii) Mathematically how would you modify the observation operator in this case such that analysis errors are uncorrelated? Comment on what this means in terms of what is now being observed.

[6 marks]

2. (a) A background error covariance matrix has the following form:

$$\mathbf{B} = \begin{pmatrix} \sigma_{\mathrm{B}}^2 & \sigma_{\mathrm{B}}^2\mu \\ \sigma_{\mathrm{B}}^2\mu & \sigma_{\mathrm{B}}^2 \end{pmatrix}, \qquad \mu > 0.$$

(i) What is the meaning of each parameter σ_{B} and μ?

[4 marks]

(ii) Work out the matrix of eigenvectors ($\mathbf{V}$) and eigenvalues ($\mathbf{\Lambda}$). You may use the formulae provided for the lectures. Each eigenvector in $\mathbf{V}$ should have unit length.

[6 marks]

(iii) Consider the following 4-D variational cost function in incremental form:

$$\begin{aligned} J[\delta\mathbf{x}] &= \frac{1}{2}\delta\mathbf{x}^{\mathrm{T}}\mathbf{B}^{-1}\delta\mathbf{x} + \\ & \frac{1}{2}\sum_{t=0}^{T}\left[\delta\mathbf{y}(t) - \mathbf{H}_t\mathbf{M}_{t\leftarrow 0}\delta\mathbf{x}\right]^{\mathrm{T}}\mathbf{R}_t^{-1}\left[\delta\mathbf{y}(t) - \mathbf{H}_t\mathbf{M}_{t\leftarrow 0}\delta\mathbf{x}\right], \end{aligned}$$

where $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_{\mathrm{B}}$ is the increment at the beginning of the time window, t is the time step $(0 \le t \le T)$,
$\delta\mathbf{y}(t) = \mathbf{y}(t) - \mathcal{H}_t(\mathcal{M}_{t\leftarrow 0}(\mathbf{x}_{\mathrm{B}}))$, $\mathbf{R}_t$ is the observation error covariance matrix, $\mathbf{y}(t)$ is the observation vector, $\mathcal{H}_t$ and $\mathcal{M}_{t\leftarrow 0}$ are the observation and model operators respectively, and $\mathbf{H}_t$and $\mathbf{M}_{t\leftarrow 0}$ are their respective Jacobians.
For the $\mathbf{B}$-matrix given at the start of Question 2(a) and using your result from part (ii), give a control variable transform, $\mathbf{U}$ that transforms a 'control space' ($\delta\boldsymbol{\chi}$, which has components whose background errors are uncorrelated and each has variance 1) to the $\delta\mathbf{x}$-space. Show the transform equation itself and the form of $\mathbf{U}$ (as matrix expressions).

[8 marks]

(iv) What is the cost function written in terms of $\delta\boldsymbol{\chi}$ and $\mathbf{U}$, that is exactly equivalent to the cost function given above? Show each step and give the result in the simplest possible form by writing $\mathbf{B}$ in eigen-form.

[7 marks]

(b) The observation update of an ensemble square root filter, at time-step k, satisfies

$$\begin{aligned} \overline{\mathbf{x}_k^a} &= \overline{\mathbf{x}_k^f} + \mathbf{K}_k(\mathbf{y}_k - \overline{\mathbf{y}_k^f}), & (1) \\ \mathbf{X}_k^a &= \mathbf{X}_k^f \mathbf{T}_k, & (2) \end{aligned}$$

where there are m ensemble members, $\overline{\mathbf{x}_k^f} \in \mathbb{R}^n$ is the mean of the forecast ensemble, $\overline{\mathbf{x}_k^a} \in \mathbb{R}^n$ is the mean of the analysis ensemble, and $\mathbf{y}_k \in \mathbb{R}^p$ is a vector of observations. The Kalman gain is given by

$$\begin{aligned} \mathbf{K}_k &= \mathbf{X}_k^f(\mathbf{Y}_k^f)^T \mathbf{D}_k^{-1}, & (3) \\ \mathbf{D}_k &= \mathbf{Y}_k^f(\mathbf{Y}_k^f)^T + \mathbf{R}_k, & (4) \\ & & (5) \end{aligned}$$

and the $m \times m$ matrix $\mathbf{T}_k$ satisfies

$$\mathbf{T}_k \mathbf{T}_k^T = \mathbf{I} - (\mathbf{Y}_k^f)^T \mathbf{D}_k^{-1} \mathbf{Y}_k^f. \quad (6)$$

(i) Describe the meaning of each of the terms $\overline{\mathbf{y}_k^f}, \mathbf{X}^f, \mathbf{X}^a, and \mathbf{Y}^f$ making sure that you give the dimension of each vector or matrix.

[10 marks]

(ii) The solution, $\mathbf{T}_k$ to equation (6) is not unique. Explain why this is the case and discuss the implications of this for the analysis ensemble mean.

[15 marks]

3. (a) Consider the standard Kalman filtering problem for $\mathbf{x} \in \mathbb{R}^3$ with a state evolution model

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

observation operator,

$$\mathbf{H} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2+\delta & 2 \end{pmatrix},$$

and covariance matrices $\mathbf{P}_0 = \mathbf{I}_{3\times 3}$ and $\mathbf{R} = \delta^2 \mathbf{I}_{2\times 2}$.

(i) What rank have $\mathbf{H}$, $\mathbf{HP}_0\mathbf{H}^T$, and $\mathbf{HP}_0\mathbf{H}^T + \mathbf{R}$ for $\delta > 0$?

[8 marks]

(ii) How would your answer to part (i) change if $\delta \to 0$?

[3 marks]

(iii) Is this system observable for $\delta > 0$ and $\delta \to 0$?

[8 marks]

(iv) What advice would you give to someone planning to implement a Kalman filter for this system for a small value of δ?

[6 marks]

(b) Several operational weather forecasting centres are now using hybrid ensemble-variational techniques for data assimilation, where a forecast ensemble is used to calculate the forecast error covariance. You have been asked to implement an Ensemble 3DVAR system with cost function

$$J(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{x}_b)^T (\mathbf{P}^f)^{-1} (\mathbf{x} - \mathbf{x}_b) + \frac{1}{2}(\mathbf{y} - \mathbf{H}(\mathbf{x}))^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}(\mathbf{x})), \tag{7}$$

where $(\mathbf{P}^f)^{-1}$ is a pseudo-inverse defined only in the ensemble subspace.

(i) Explain some of the problems you would expect the method to encounter when using a small ensemble relative to the size of the state-space.

[8 marks]

(ii) Explain how you would set up an identical twin experiment to test your implementation of the method. Be sure to include a description of how you would set up forecast-assimilation cycling for this assimilation scheme. You need NOT describe the metrics you would use to evaluate the results.

[17 marks]

[End of Question Paper]