Candidates are admitted to the examination room ten minutes before the start of the examination. On admission to the examination room, you are permitted to acquaint yourself with the instructions below and to read the question paper.

Do not write anything until the invigilator informs you that you may start the examination. You will be given five minutes at the end of the examination to complete the front of any answer books used.

April 2014 **MTMD02**

Answer Book
Data Sheet
Any bilingual English language dictionary permitted
Only Casio-fx83 calculators are permitted

UNIVERSITY OF READING

Operational data assimilation techniques (MTMD02)

Two hours

Answer **ANY TWO** questions

The marks for the individual components of each question are given in [ ] brackets. The total mark for the paper is 100

1. Weak-constraint 4D-VAR (WC4D-VAR) is often considered a 'gold-standard' method of estimating the evolving state of the weather.

(a) Give three assumptions that are made in the application of WC4D-VAR to the real world.

[3 marks]

(b) A simplification that is often made to WC4D-VAR for practical purposes is called strong-constraint 4D-VAR (SC4D-VAR).

(i) What is neglected in the basic SC4D-VAR cost function compared to WC4D-VAR?

[2 marks]

(ii) Write down the incremental form of the SC4D-VAR cost function with argument $\delta\mathbf{x}$, the increment in model space at the initial time. Explain the meaning of all symbols that you use and note which operators can be non-linear and which are linear.

[5 marks]

(iii) Derive an expression for the gradient vector (with respect to $\delta\mathbf{x}$).

[5 marks]

(c) Finding the maximum a-posteriori (MAP) is one way of defining the analysis.

(i) What is the mathematical meaning of the MAP?

[3 marks]

(ii) Assuming linearity of operators and that the background state is used as the reference state, show how the following explicit expression for the MAP analysis, $\mathbf{x}_{\text{MAP}}$, is found from the gradient expression:

$$\mathbf{x}_{\text{MAP}} = \mathbf{x}_{\text{B}} + \left(\mathbf{B}^{-1} + \sum_{t=0}^{T} \mathbf{M}_{t\leftarrow 0}^{\text{T}}\,\mathbf{H}_t^{\text{T}}\mathbf{R}_t^{-1}\mathbf{H}_t\mathbf{M}_{t\leftarrow 0}\right)^{-1} \times \sum_{t=0}^{T} \mathbf{M}_{t\leftarrow 0}^{\text{T}}\,\mathbf{H}_t^{\text{T}}\mathbf{R}_t^{-1}\left[\mathbf{y}_t - \mathbf{h}_t\left(\mathcal{M}_{t\leftarrow 0}(\mathbf{x}_{\text{B}})\right)\right],$$

where the notation is standard and where state vectors are for t = 0 (use of different symbols is acceptable, but your result should follow from (b)).

[8 marks]

(d) A further simplification to SC4D-VAR is called 3D-FGAT, which stands for 3D-VAR First Guess at Appropriate Time. In a similar way to part (b), write down the following for 3D-FGAT.

(i) The cost function.

[2 marks]

(ii) The gradient.

[2 marks]

(iii) The MAP analysis.

[2 marks]

(e) The MAP analysis equations in 1(c) are to be applied to a system where one variable is to be estimated. There are two independent observations (the first is at t = 0 and the second is at t = 2). The following characteristics apply (assume dimensionless units):

$\mathbf{B} = (1)$, $\mathbf{h}_t(\mathbf{x}_t) = \mathbf{x}_t$, $\mathbf{x}_B = (125/4)$, $\mathbf{x}_t = \mathcal{M}_{t\leftarrow t-1}(\mathbf{x}_{t-1}) = 4\mathbf{x}_{t-1}/5$, $\mathbf{R}_t = (2)$, $\mathbf{y}_0 = (31)$ and $\mathbf{y}_2 = (22)$.

(i) Give numerical values for the following quantities $\mathbf{M}_{1\leftarrow 0}$, $\mathbf{M}_{2\leftarrow 1}$, $\mathbf{H}_0$, and $\mathbf{H}_2$.

[4 marks]

(ii) Give numerical values for $\mathbf{y}_0 - \mathbf{h}_0\left(\mathcal{M}_{0\leftarrow 0}(\mathbf{x}_\mathrm{B})\right)$ and $\mathbf{y}_2 - \mathbf{h}_2\left(\mathcal{M}_{2\leftarrow 0}(\mathbf{x}_\mathrm{B})\right)$, each for SC4D-VAR and 3D-FGAT.

[4 marks]

(iii) Give numerical values for $\mathbf{x}_{\mathrm{MAP}}$ for SC4D-VAR and 3D-FGAT.

[10 marks]

2. (a) Hybrid techniques are widely used for operational data assimilation where the $n \times n$ background error covariance matrix in 4D-VAR, $\mathbf{B}$, is replaced by $\mathbf{B}^H = \alpha\mathbf{B} + (1-\alpha)\,\mathbf{P}^f$, where $\mathbf{P}^f$ is the background error covariance matrix derived from an N-member ensemble, and α is a tuning parameter ($0 \le \alpha \le 1$). Let $\mathbf{X}$ be the $n \times N$ matrix whose columns are the N ensemble perturbations.

(i) Write down the matrix relationship between $\mathbf{P}^f$ and $\mathbf{X}$, assuming that no localization method is used.

[3 marks]

(ii) What are the main reasons for using this hybrid matrix instead of pure $\mathbf{B}$?

[2 marks]

(iii) Why not use just the matrix $\mathbf{P}^f$ in operational data assimilation?

[2 marks]

(iv) For a test system, $\mathbf{B}$ and $\mathbf{X}$ are the following:

$$\mathbf{B} = \begin{pmatrix} 9 & 5 & 1 \\ 5 & 25 & 6 \\ 1 & 6 & 25 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 3 & 1 & -4 \\ 3 & 2 & -5 \\ 2 & 1 & -3 \end{pmatrix}.$$

Write down N, n, and the matrices $\mathbf{P}^f$ and $\mathbf{B}^H$ when $\alpha = 0.5$.

[8 marks]

(v) Using the numerical values of $\mathbf{B}$, $\mathbf{X}$ and α defined above, write down algebraically, for a system with 3 variables and 3 ensemble members, the following (you may assume that $\mathbf{B}$ may be written as $\mathbf{B} = \mathbf{U}\mathbf{U}^T$):

(A) An uncorrelated hybrid control vector that could be used in a hybrid data assimilation system.

[4 marks]

(B) The particular control variable transform for this system that has an implied background error covariance matrix which matches $\mathbf{B}^{H}$ defined in this question. Note that for the $\mathbf{B}$ specified above, $\mathbf{U}$ is (approximately):

$$\mathbf{U} = \begin{pmatrix} 3.00 & 0.00 & 0.00 \\ 1.67 & 4.71 & 0.00 \\ 0.33 & 1.15 & 4.85 \end{pmatrix}.$$

[6 marks]

(b) The perturbed observation ensemble Kalman filter (EnKF) observation update equation at time t, is given by

$$x_i^a = x_i^f + K\left(y - Hx_i^f + \varepsilon_i\right), \tag{2.1}$$

where $\left\{x_i^f \in \mathbb{R}^n, i = 1,2,\ldots,m\right\}$ is an ensemble of forecasts, $y \in \mathbb{R}^p$ is a vector of observations, and the Kalman gain may be written

$$K = P^f H^T\left(HP^f H^T + R\right)^{-1}. \tag{2.2}$$

(i) Describe the meaning of each of the terms ε_i, P^f, H, and R, making sure that you give the size of each vector or matrix.

[8 marks]

(ii) Suppose that you are applying the perturbed observation EnKF to a scalar problem ($n = 1$), with a perfect forecast model,

$$x(t+1) = \alpha x(t), \tag{2.3}$$

and direct observations with an observation error variance of 1. You are told that the true initial forecast error distribution is Gaussian with mean 0, and variance 1. You are also given a starting ensemble of $\{-0.5, 0.3, 0.5\}$.

Calculate the ensemble forecast error variance (without inflation) after one time-step.

[3 marks]

(iii) Find the optimal inflation factor by minimizing the difference between the true forecast error variance and the ensemble forecast error variance (with inflation).

[6 marks]

(iv) Is the inflation factor that you found also optimal for subsequent time-steps? Explain your answer.

[8 marks]

3. (a) Consider the dynamical system

$$\begin{pmatrix} u(t+1) \\ v(t+1) \\ w(t+1) \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}. \tag{3.1}$$

At each time, t, the following observations are made,

$$y(t) = \begin{pmatrix} u(t) \\ u(t) + v(t) \end{pmatrix}. \tag{3.2}$$

A Kalman filter applied to this system suffers from filter divergence.

What does filter divergence mean?

Using the concept of observability explain why the filter diverges and give an example of what you could do to stabilize the system.

[12 marks]

(b) An ensemble transform Kalman filter (ETKF) with 100 ensemble members is applied to a weather forecasting problem. The observation increments are affecting regions geographically located a long distance from the observation sites. A meteorologist tells you that this does not seem physically realistic.

(i) Describe why the unphysical observation increments may be occurring.

[8 marks]

(ii) Describe a technique that you could use to reduce the unphysical observation increments and comment on its mathematical properties. (You should describe the key ideas of the technique but you need not give the full implementation for the ETKF).

[10 marks]

(c) Several operational weather forecasting centres are now using hybrid ensemble-variational techniques for data assimilation, where a forecast ensemble is used to calculate the forecast error covariance. For example the cost function for Ensemble-3DVAR is given by

$$J(x) = \frac{1}{2}(x - x_b)^T (P^f)^{-1}(x - x_b) + \frac{1}{2}\big(y - H(x)\big)^T R^{-1}\big(y - H(x)\big),$$

where $\left(P^f\right)^{-1}$ is a pseudo-inverse defined only in the ensemble subspace.

Compare and contrast the hybrid approach with ensemble Kalman filter methods.

[20 marks]

(End of Question Paper)